# International school on Algebraically Geometry and Algebraic Groups

(Hanoi, November 2021)

## Commutative Algebra, November 01, 2021

**Exercise 1** Let $A$ be a commutative ring and $A[[x_1, x_2, ..., x_n]]$ be the ring of formal power series. Let $f = f_0 + f_1 + \cdots \in A[[x_1, x_2, ..., x_n]]$ where $f_i$ is a homogeneous polynomial of degree $i$.

1) Prove that $f$ is a unit $A[[x_1, x_2, ..., x_n]]$ if and only if $f_0$ is a unit in $A$.

2) $k[[x]]$ is a PID, where $k$ is a field.

3) Prove that if $k$ is algebraically closed, $f$ is a unit in $k[[x]]$. Then, for all $m \geq 1$, there is $g \in k[[x]]$ such that $g^m = f$.

**Exercise 2** Let $A$ be a Noetherian ring. Prove that the ring of formal power series $A[[x]]$ is Noetherian. Deduce that $k[[x_1, x_2, ..., x_n]]$ is Noetherian, where $k$ is a field.

**Exercise 3** Let $p$ is a prime number and

$$E(p) = \{\alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^n} + \mathbb{Z} \text{ for some } r \in \mathbb{Z}, n \in \mathbb{N}\}.$$

For each $t \in \mathbb{N}$, set

$$G_t = \{\alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^t} + \mathbb{Z} \text{ for some } r \in \mathbb{Z}\}.$$

Prove that

1) $G_t$ is the submodule of $E(p)$ generated by $\dfrac{1}{p^t} + \mathbb{Z}$, for $t \in \mathbb{N}$.

2) Each proper submodule of $E(p)$ equal to $G_i$ for some $i \in \mathbb{N}$.

3) We have

$$G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots$$

and $E(p)$ is an Artinian, non Noetherian $\mathbb{Z}$-module.

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**Exercise 4** Let $R$ be a commutative ring and $M$ be a finitely generated $R$-module. Let $\alpha : M \to M$ be an endomorphism of $M$. Assume that $\alpha$ is surjective. Prove that $\alpha$ is an isomorphism.

**Exercise 5** Let $R = \mathbb{C}[x^4, x^3y, xy^3, y^4]$. Find the integral closure of $R$.

**Exercise 6** Find a Noetherian normalization of the following $\mathbb{C}$-algebra

$$R = \mathbb{C}[x, y, z]/(xy + z^2, x^2y - xy^2 + z^4 - 1).$$

**Exercise 7** Find all maximal ideals of the polynomial ring $\mathbb{R}[x, y]$.