International school on Algebraically Geometry and Algebraic Groups

(Hanoi, November 2021)

Commutative Algebra, November 01, 2021

Exercise 1 Let A be a commutative ring and $A[[x_1, x_2, ..., x_n]]$ be the ring of formal power series. Let $f = f_0 + f_1 + \cdots \in A[[x_1, x_2, ..., x_n]]$ where f_i is a homogeneous polynomial of degree i.

1) Prove that f is a unit $A[[x_1, x_2, ..., x_n]]$ if and only if f_0 is a unit in A.

2) $k[[x]]$ is a PID, where k is a field.

3) Prove that if k is algebraically closed, f is a unit in $k[[x]]$. Then, for all $m \geq 1$, there is $g \in k[[x]]$ such that $g^m = f$.

Exercise 2 Let A be a Noetherian ring. Prove that the ring of formal power series $A[[x]]$ is Noetherian. Deduce that $k[[x_1, x_2, ..., x_n]]$ is Noetherian, where k is a field.

Exercise 3 Let p is a prime number and

$$E(p) = \{\alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^n} + \mathbb{Z} \text{ for some } r \in \mathbb{Z}, n \in \mathbb{N}\}.$$

For each $t \in \mathbb{N}$, set

$$G_t = \{\alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^t} + \mathbb{Z} \text{ for some } r \in \mathbb{Z}\}.$$

Prove that

1) G_t is the submodule of $E(p)$ generated by $\dfrac{1}{p^t} + \mathbb{Z}$, for $t \in \mathbb{N}$.

2) Each proper submodule of $E(p)$ equal to G_i for some $i \in \mathbb{N}$.

3) We have

$$G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots$$

and $E(p)$ is an Artinian, non Noetherian $\mathbb{Z}$-module.

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Exercise 4 Let R be a commutative ring and M be a finitely generated R-module. Let $\alpha : M \to M$ be an endomorphism of M. Assume that α is surjective. Prove that α is an isomorphism.

Exercise 5 Let $R = \mathbb{C}[x^4, x^3y, xy^3, y^4]$. Find the integral closure of R.

Exercise 6 Find a Noetherian normalization of the following $\mathbb{C}$-algebra

$$R = \mathbb{C}[x, y, z]/(xy + z^2, x^2y - xy^2 + z^4 - 1).$$

Exercise 7 Find all maximal ideals of the polynomial ring $\mathbb{R}[x, y]$.

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Exercise 8 Let p be a prime number and n be a positive integer. Compute $\ell_{\mathbb{Z}}(\mathbb{Z}/p^n\mathbb{Z})$. Deduce that $\ell_{\mathbb{Z}}(\mathbb{Z}/20\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z})$.

Exercise 9 Let $R = k[[x, y]]$ be a ring of formal power series with maximal ideal $\mathfrak{m}$. For each $n \geq 0$, compute $\ell_R(R/I^{n+1})$. Deduce the Hilbert coefficients R w.r.t I in the following cases:

a) $I = \mathfrak{m}$.

b) $I = (x^2, xy^2, y^3)$.

c) $I = (x^2, y^3)$.

Exercise 10 Let $R = k[[t^3, t^5, t^7]]$ and $I = (t^3, t^5)$. Compute $\ell_R(R/I^{n+1})$.

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Exercise 11 Let $R = k[x, y]/(y^2 - x^3)$. Prove that $R_{(x,y)}$ is not a regular local ring (RLR).

Exercise 12 Let $S = k[x, y, z]$ be a polynomial ring, $P = (x, y, z)S$. Set $R = S_P/(z^2 - xy)S_P$. Prove that R is not a RLR but $R_{\mathfrak{p}}$ is a RLR for every $\mathfrak{p} \in Spec(R) \setminus \{\mathfrak{m}\}$, where $\mathfrak{m}$ denotes the maximal ideal of R.

Exercise 13 Let $(S, \mathfrak{n})$ be a discrete valuation ring and let $\mathfrak{n} = yS$. We set $R = S[[x]]/(x^n - y)$ where $S[x]$ denotes the polynomial ring and $n \geq 1$ an integer. Show that R is a RLR.

Exercise 14 Compute the graded minimal free resolution of R/I. Deduce the Hilbert series of R/I in the following cases:

a) Let $R = k[x, y]$ and $I = (x^3, xy, y^5)$.

b) Let $R = k[x, y, z, u]$ and $I = (x^3, xy, yz)$.