# Singularity in Algebraic Geometry

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# Getting Start

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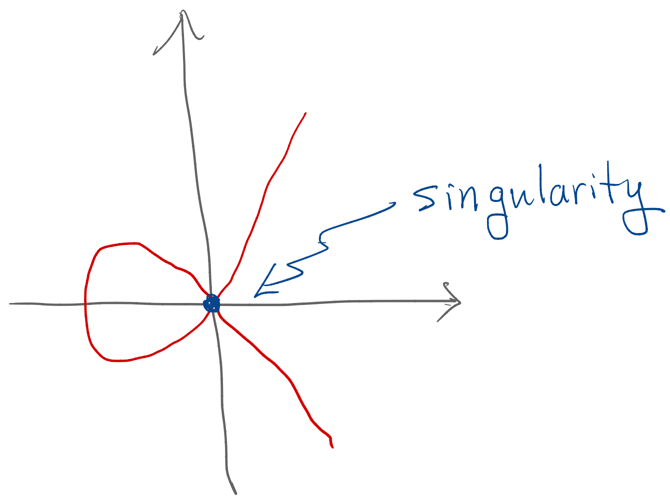
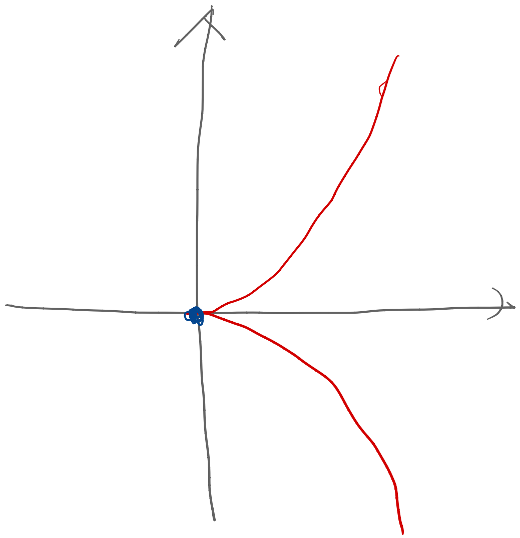



$C_2$ $\quad y^2 - x^3$



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$o \in X$ *is singular if*

$$\frac{\partial f}{\partial x_i}|_o = 0, \forall i.$$

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*Suppose that* $X = (f_1 = f_2 = ... = f_r = 0) \subset \mathbb{A}^n$ *is an affine variety of dimension m.*
$X$ *is singular at* $o \in X$ *if*

$$rk\Big((\frac{\partial f_j}{\partial x_i}|_o)_{ij}\Big) < n - m.$$

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3. $E^2 = -1$;
4. $K_Y = \pi^* K_{\mathbb{A}^2} + E$.

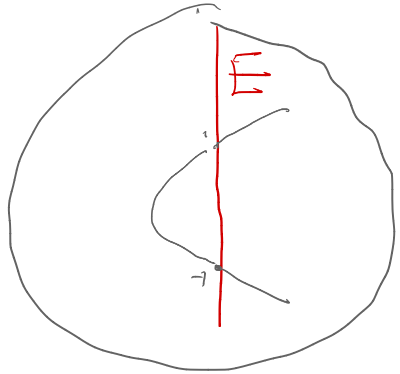
$$Y \subset \mathbb{A}^2 \times \mathbb{P}^1$$
$$(x,y) \quad [z_0, z_1]$$

$$\downarrow$$

$$\mathbb{A}^2$$

$$Y = V(xz_1 - yz_0)$$

or $Y = \{ (x,y), [z_0, z_1] \mid (x:y) = (z_0 : z_1) \}$




$E \cong \mathbb{P}^1$

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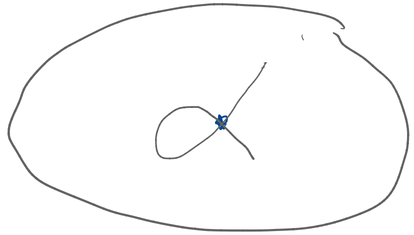

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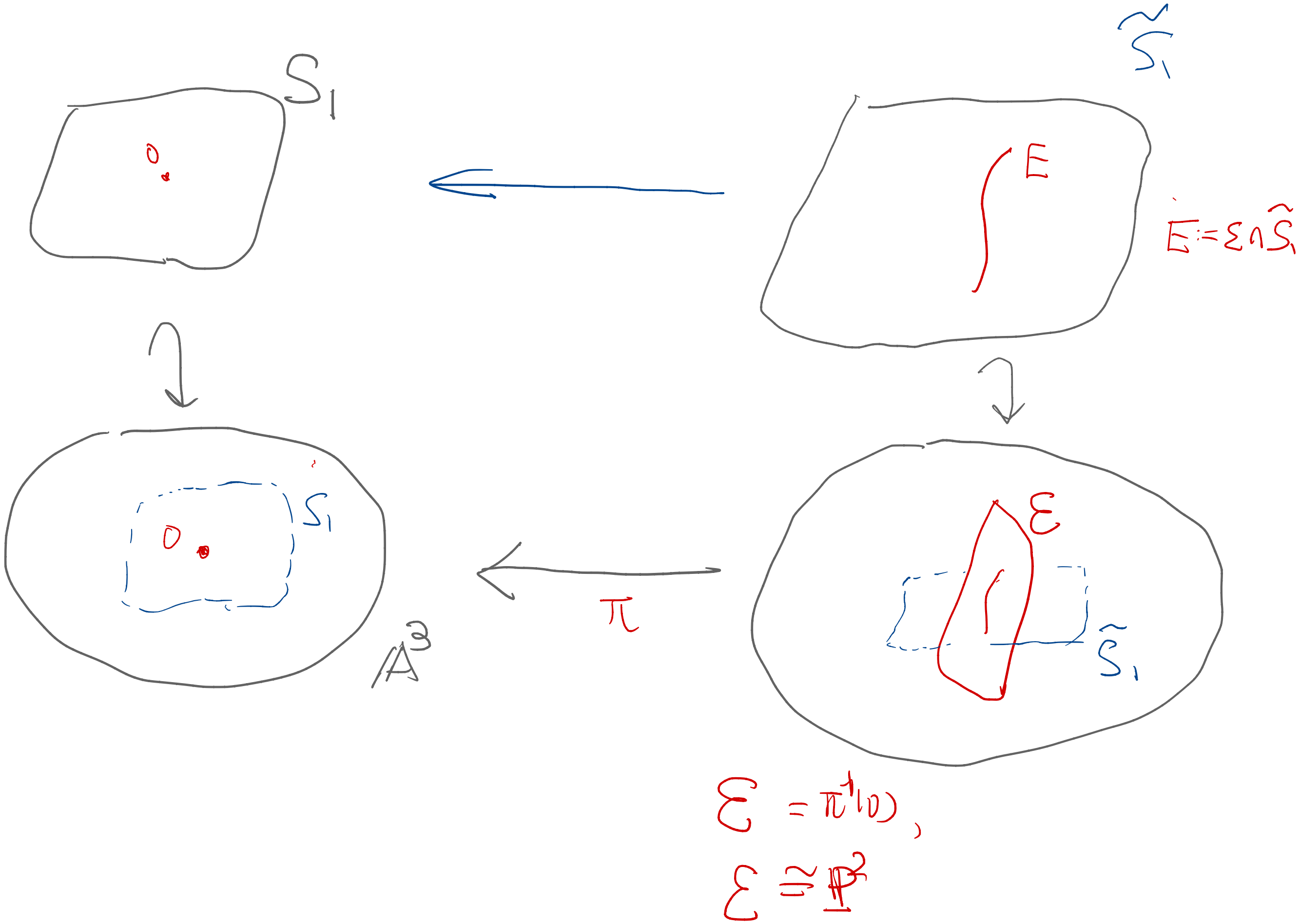
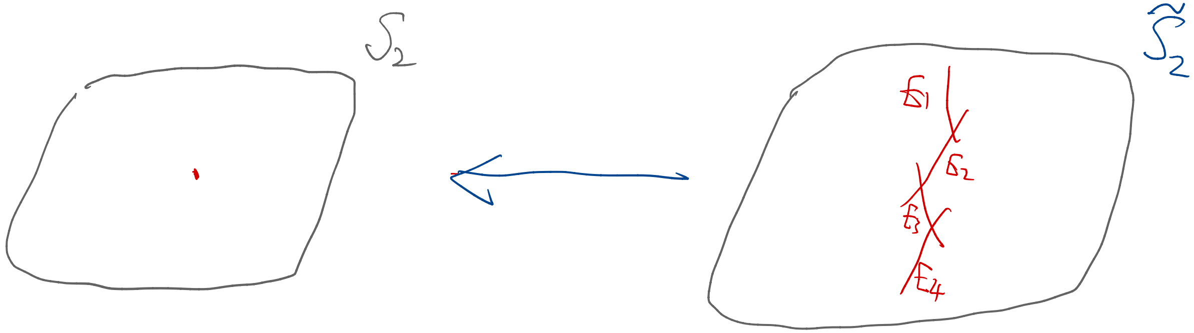
3. $S_3$ *defined by* $f(x, y) = x^3 + y^3 + z^3$*.*

$S_1: \quad x^2+y^2+z^2$

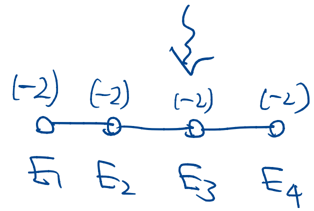


## $S_2$

$x^2+y^2+z^5$




$$\begin{cases} E_i \cong \mathbb{P}^1 \\ E_i^2 = -2 \\ E_i \cdot E_j = 1 \text{ or } 0 \end{cases}$$




$A_4$

## $S_3$

$x^3+y^3+z^3$


$E \cong$ elliptic curve (cubic in $\mathbb{P}^2$)

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### Theorem (Mori)

*Let $X$ be a smooth projective threefold. There is a sequence of birational maps*

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However, we need to allow "mild" singularities (terminal singularities indeed) in order for the above program to work.

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The $o \in X$ is terminal if and only if $o \in X$ is non-singular.

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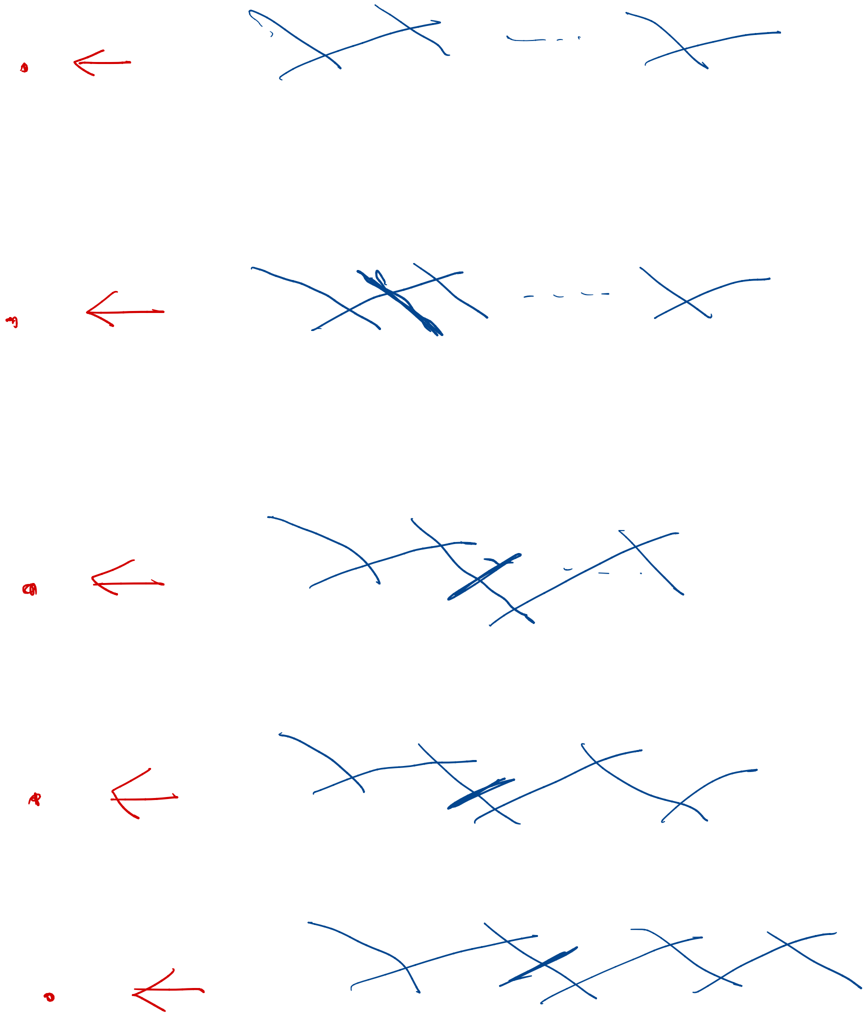
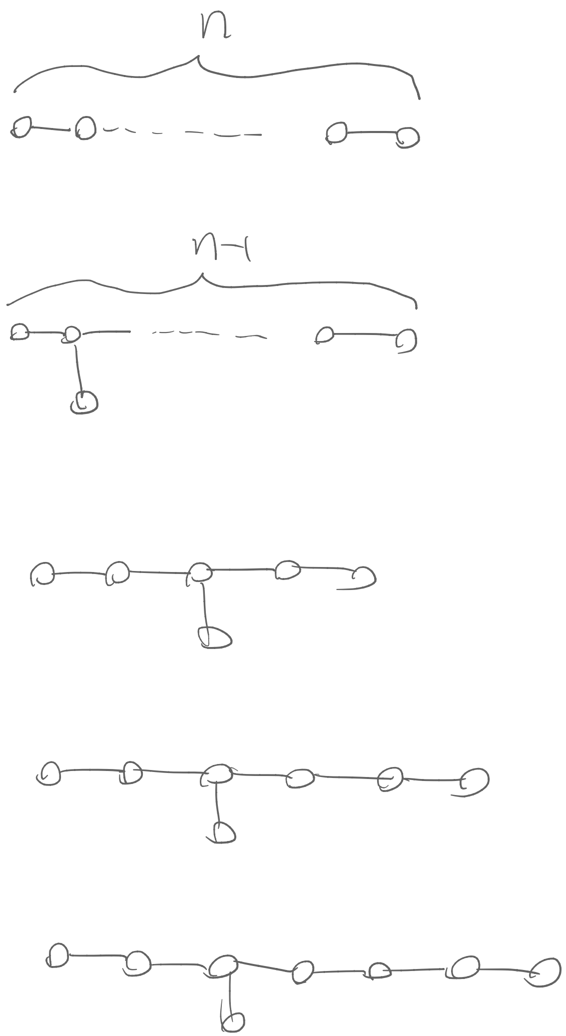
The following are equivalent:

# A–D–E

Exc(π)






$A_n$

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5. $cD/2$: $P \in X$ is given by $(\varphi = 0) \subset \mathbb{C}^4/\frac{1}{2}(1, 1, 0, 1)$ with $\varphi$ being certain $cD$ type.

# Threefold Terminal Singularities

$o \in X$ is a quotient of a smooth or an isolated cDV point.

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7. $cE/2$: $(x^2 + y^3 + yg(z, u) + h(z, u) = 0) \in \mathbb{C}^4/\frac{1}{2}(1, 0, 1, 1)$.

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Thank you!