International School on Algebraic geometry and Algebraic Groups
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Lectures on Algebraic Varieties

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1. Varieties, defining ideals and coordinate ring

What is Algebraic Geometry?

- Descartes: A geometric point is determined by its coordinate in a coordinate system. Coordinates are tuples of numbers.
- A geometric object is a set of points which is set of tuples of numbers satisfying some properties or some numeric conditions.
- A geometric object in Algebraic Geometry is a set of points, whose coordinates satisfy a system of algebraic equations. This marks the difference between Algebraic Geometry and Analytic Geometry or Differential Geometry, ...

Example 1.1. Examples of objects in Algebraic Geometry:

- A line on the plane: $ax + by + c = 0$.
- A circle on the plane: $x^2 + y^2 = r$.
- An ellipse, a parabola, a hyperbola, ...
- In general, a system of algebraic equations on a set of <u>indeterminates</u>.
- Inequalities are $\boxed{NOT}$ allowed!!!

IS THIS NOT TWO STRICT?

DOES IT MAKE SENSE TO STUDY THESE GEOMETRIC OBJECTS???

There are more reasons for Algebraic Geometry than just the above geometric objects.

MAIN POINT: Which value set should the solution of the system of algebraic equations belong to?

Because we consider algebraic equations, we can take any value set, as long as one can perform on them addition and multiplication! ★

Example 1.2. Consider the famous equation in the integers

$$x^2 + y^2 = z^2.$$

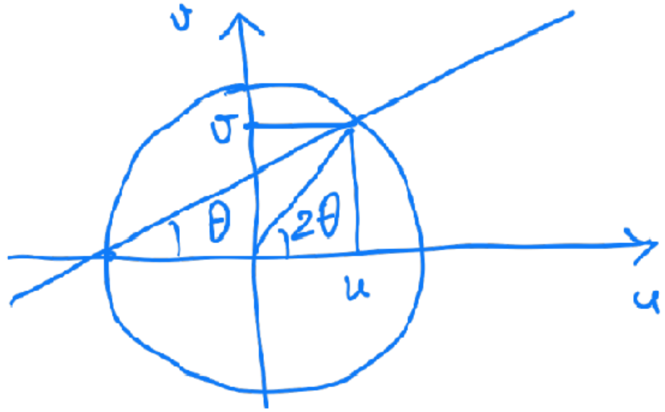
We will assume $xyz \neq 0$. Divide both sides by z we are led to

$$(\frac{x}{z})^2 + (\frac{y}{z})^2 = 1. \tag{1}$$

Setting $u = \dfrac{x}{z}; v = \dfrac{y}{z}$, we have

$$u^2 + v^2 = 1. \tag{2}$$

This last equation determined a circle on the coordinate plane.



Draw a line through the point $(-1, 0)$ as shown in the picture, with slope θ, intersecting the circle at point $P(u, v)$. Then

$$\tan\theta = \frac{v}{u+1}. \tag{3}$$

Elementary trigonometry shows that

$$\begin{cases} u = \cos 2\theta \\ v = \sin 2\theta. \end{cases} \tag{4}$$

Thus we have:

★ An integral solution of (1) with x, y, z coprime is the same as a rational solution of (2).

★ A rational solution of (2) is given by in terms of a rational number $\tan\theta$ as in (3).

The geometric consideration of the set of real of (2) helps solving (1).

The example above is an illustration for the claim that Algebraic Geometry is a bridge between different branches of Mathematics such as Number Theorey, Geometry, Analysis, and also Physic, Informatics,... The basic language of Algebraic Geometry is Algebra, especially Commutative Algebra.

1.1. **Affine spaces and affine algebraic sets.** Affine geometry, in contrast with Euclidean geometry, ignores the metrics notion of distance and angle. The main notion that remains is that of parallel lines.

We shall consider solution set of systems of polynomial equations in an algebraically closed field.

Question 1.3. Why is an algebraically closed field?

To ensure the existence of all possible solutions - we want to see the "geometric nature" of the problem.

See Hilbert Nullstelensatz below.

Fix a field k which is an algebraically closed: any polynomial with coefficients from k and of degree at least one has at least one root in k.

Example 1.4. The field of complex numbers $\mathbb{C}$. (In what follows we are interpret k as $\mathbb{C}$, BUT shall mainly consider only REAL solutions - that is to interpret the REAL part of the geometric pictures).

A system of polynomial equations is given by polynomials $f_1, f_2, ..., f_r, ... \in k[z_1, ..., z_n]$ as follows

$$\begin{cases} f_1 = 0 \\ f_2 = 0 \\ \quad ... \\ f_r = 0 \\ \quad ... \end{cases} \tag{5}$$

A theorem of Hilbert ensures however that this system is equivalent (i.e., has the same set of solutions) to a system with finitely many equations. (However, Hilbert's theorem does not tell us which system it is).

Theorem 1.5 (Hilbert's Basis Theorem). *Any ideal in the ring $k[z_1, ..., z_n]$ is finitely generated.*

To apply this theorem to the above system, we just notice that the solution set of (5) is the same as the set of solutions of the system

$$\{f = 0, f \in \mathfrak{I} = (f_1, f_2, ..., f_r, ...)\},$$

where $\mathfrak{I} = (f_1, f_2, ..., f_r, ...)$ is the ideal in $k[z_1, ..., z_n]$ generated by $f_1, f_2, ..., f_n, ...$.

Thus from now on, we shall have in mind the solution set of ideals in $k[z_1, z_2, ..., z_n]$, which we know that they are all finitely generated. Such a set is called "an affine algebraic set".

Example 1.6. ★ If we take ideal (0), the whole space $k^n = \{(a_1, ..., a_n) : a_i \in k\}$ is the solution set. This space will be denoted by $\mathbb{A}_k^n$.

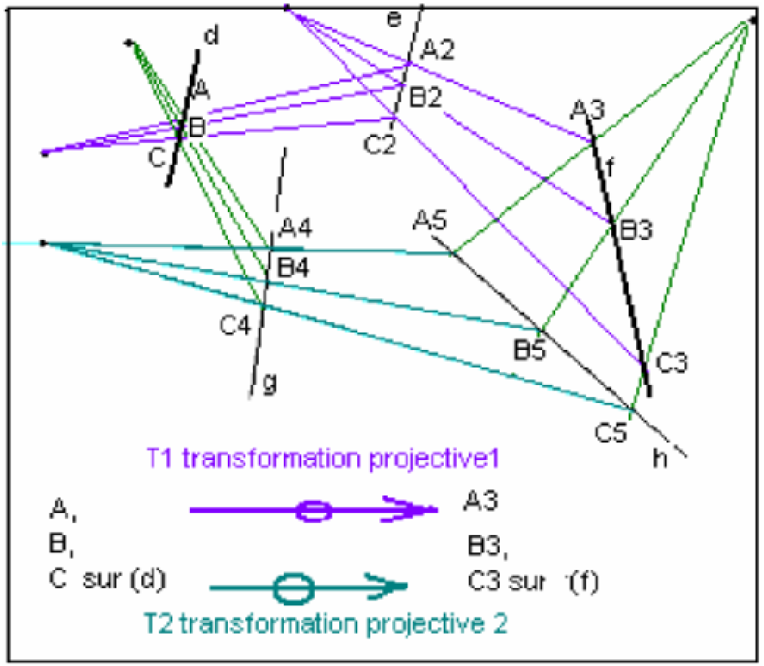
★ If we take the whole ring $R = k[z_1, ..., z_n]$ as the ideal, the solution set is empty.

★ A singe point $P(a_1, ..., a_n)$ is the solution set of the maximal ideal

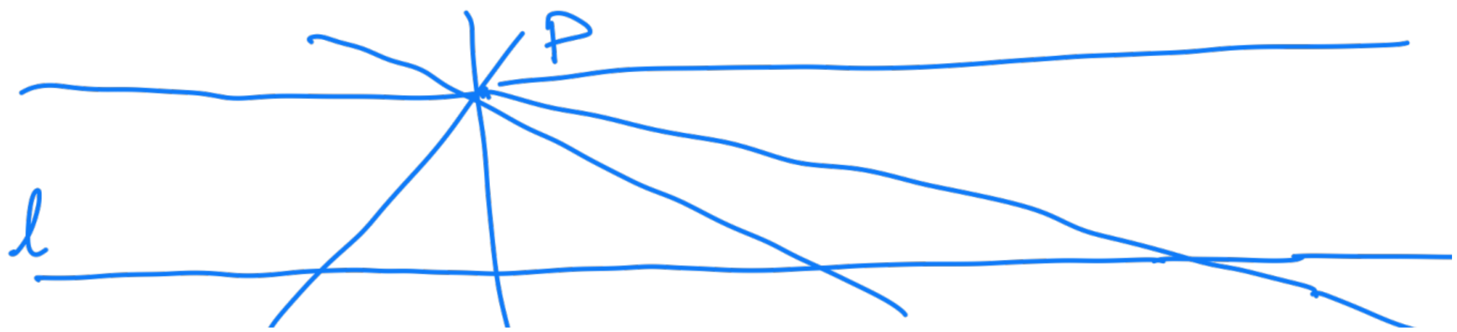
$$\mathfrak{m}_P = (z_1 - a_1, ..., z_n - a_n).$$

1.2. **Projective spaces and projective algebraic sets.** In contrast to Affine Geometry, Projective Geometry does not even have the notion of parallel. It can be seen as the geometry of constructions with straight-edge.

Models of Projective spaces are provided by Affine spaces added "points at infinity".

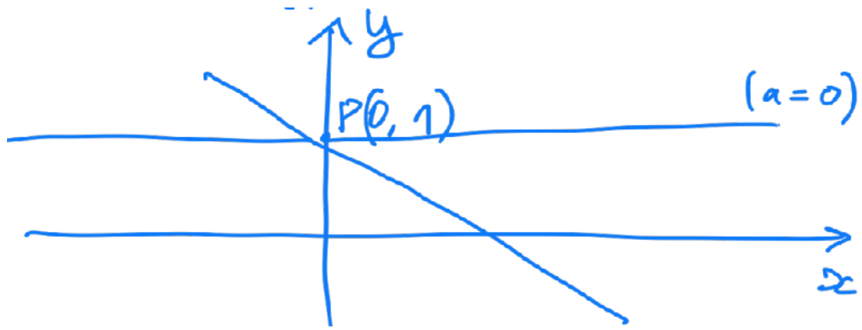


Example 1.7. $\mathbb{P}^1_k$, for $k = \mathbb{R}$: the real projective line. It is given by adding on more points at infinity.



Fix a point P outside a line $\mathbf{l}$ on a plane. Each point on l determines a line through P, an vice-versa, up to one line: the line passing through P and parallel with l certainly does not correspond to any point on l. We imagine it corresponds to a point at infinity.

Algebraically, this work for any field k. On the coordinate plane (Oxy) with $P(0, 1)$ and $l = (Ox)$.



A line Δ through $P(0, 1)$ is given by an equation

$$\Delta : ax + by = b, (a, b) \neq (0, 0).$$

Note that the pair (a, b) is determined up to a non-zero constant. Therefore instead of a pair (a, b) we consider the ratio $[a : b]$.

- If $a \neq 0$, Δ intersects with (Ox) in the point $x = \dfrac{b}{a}$.
- If $a = 0$, Δ is the line parallel to (Ox).

Thus, $[0 : b]$ corresponds to the point at infinity of the projective line $\mathbb{P}^1_k$.

The projective space $\mathbb{P}^n_k$, as a set, consists of $n-$tuples

$$\{[a_0 : a_1 : \ldots : a_n] | a_i \in k, \text{ of "ratio" not all equal to } 0.\}$$

Hence, by "ratio" we mean that two tuples $[a_0 : a_1 : ... : a_n]$ and $[\lambda a_1 : \lambda a_2 : ... : \lambda a_n]$ are the same, for all $\lambda \in k, \lambda \neq 0$.

Exercise 1.8. *Give the precise model of $\mathbb{P}^2_{\mathbb{R}}$ as the set of line in $\mathbb{R}^3$ passing through $P(0,0,1)$ and intersecting plane (Oxy) to shows that $\mathbb{P}^2_{\mathbb{R}}$ is obtained from $\mathbb{R}^2$ by adding a whole projective line at infinity.*

1.3. **Algebraic sets in $\mathbb{P}^n_k$.** The tuple $[\lambda a_0 : \lambda a_1 : ... : \lambda a_n]$ is called the homogeneous coordinates of a point in $\mathbb{P}^n_k$. We can thus note evaluate a polynomial $f \in k[Z_0, Z_1, ..., Z_n]$ at such a point. However, if f is homogeneous of degree d, i.e.,

$$f(\lambda Z_0, \lambda Z_1, ..., \lambda Z_n) = \lambda^d f(Z_0, Z_1, ..., Z_n),$$

we can speak about f "vanishing at a point" in $\mathbb{P}^n_k$, namely, if

$$f(a_0, a_1, ..., a_n) = 0.$$

Hence, we can speak about solution sets of a system of homogeneous equations.

Definition 1.9. An algebraic set in $\mathbb{P}^n_k$ is by definition the solution set of a system of homogeneous equations or equivalently the common zeros of a homogeneous ideal.

Recall: An ideal $\mathfrak{I}$ in $k[Z_0, Z_1, ..., Z_n]$ is said to be homogeneous if satisfies: $f \in \mathfrak{I}$ implies each homogeneous component of f is in $\mathfrak{I}$.

Exercise 1.10. *Show that $\mathfrak{I}$ is homogeneous if and only if it is generated by homogeneous polynomials.*

Example 1.11. Linear (sub)spaces in $\mathbb{P}_k^n$ are given by system of homogeneous equations

$$\mathbf{V} = \{ \sum_{i=0}^{n} a_{ij} Z_i, j = 1, 2, ..., d.\}$$

1.4. **Relationships between affine and projective spaces.** Consider the quotient map

$$\begin{array}{ccc} \mathbb{A}_k^{n+1} \setminus \{0\} & \longrightarrow & \mathbb{P}_k^n; \\ (a_0, a_1, ..., a_n) & \mapsto & [a_0 : a_1 : ... : a_n]. \end{array}$$

The preimage of this surjective map at a point

$$P[a_0 : a_1 : \ldots : a_n] \in \mathbb{P}_k^n$$

is the (punched) line

$$l_P = \{(\lambda a_0, \lambda a_1, \ldots, \lambda a_n) | \lambda \neq 0\}$$

in $\mathbb{A}_k^{n+1} \setminus \{0\}$. This is the generalizaton of the projective map considered before.

More general, the preimage of a linear space

$$\mathbf{V} = \{ \sum_{i=0}^{n} a_{ij} Z_i, j = 1, 2, ..., d\}$$

in $\mathbb{P}_k^n$ is the vector subspace $\widehat{\mathbf{V}} = \{ \sum_{i=0}^{n} a_{ij} Z_i, j = 1, 2, ..., d.\} \setminus \{0\}$ in $\mathbb{A}_k^{n+1} \setminus \{0\}$.

Exercise 1.12. *Check this and use it to prove the following claim: Two linear spaces* $\mathbf{V}$ *and* $\mathbf{W}$ *in* $\mathbb{P}_k^n$ *intersect non-trivially if and only if their preimages*

$$\widehat{\mathbf{V}} \cap \widehat{\mathbf{W}} \neq 0.$$

Consequently, if

$$\dim \widehat{\mathbf{V}} + \dim \widehat{\mathbf{W}} \geq n + 2$$

then $\mathbf{V}$ and $\mathbf{W}$ intersect non-trivially in $\mathbb{P}_k^n$. We set

$$\dim \mathbf{V} := \dim \widehat{\mathbf{V}} - 1.$$

The the claim above amounts to saying: if

$$\dim V + \dim W \geq n$$

then V and W intersect non-trivially in $\mathbb{P}_k^n$.

In particular, any two lines in the projective plan intersect non-trivally.

On the other hand, $\mathbb{P}^n_k$ can be seen a comprising of affine subsets, determined by the condition:

$$U_i = \big\{[a_0 : a_1 : ... : a_n] | a_i \neq 0\big\},$$

for each i fixed.

There is a natural bijection $\varphi_i : U_i \longrightarrow \mathbb{A}^n_k$, given by

$$[a_0 : a_1 : ... : a_i : ... : a_n] \mapsto (\frac{a_0}{a_i}, \frac{a_1}{a_i}, ..., \widehat{1}, ..., \frac{a_n}{a_i}),$$

where $1 = \dfrac{a_i}{a_i}$ is omitted.

Exercise 1.13. *Show that if $X \subset \mathbb{P}^n_k$ is an algebraic set then $X \cap U_i$ is an affine algebraic set in $U_i = \mathbb{A}^n_k$. Show that the converse is also true: $X \subset \mathbb{P}^n_k$ is an algebraic set if $X \cap U_i$ is an affine algebraic set in U_i each $i = \overline{0, n}$.*

1.5. **Affine and Projective varieties.** Assume that $Z \subset \mathbb{A}^n_k$ is an algebraic set. Thus Z is the solution set of an ideal $\mathfrak{I}$. Let $I(Z)$ denote the set of all polynomials in $k[z_1, ..., z_n]$ vanishing on Z. Then, by definition

$$\mathfrak{I} \subset I(Z).$$

It turns out that $I(Z)$ is the radical of $\mathfrak{I}$.

Recall: The radical of $\mathfrak{I}$ in $k[z_1, ..., z_n]$ is the set of all polynomials in $k[z_1, ..., z_n]$ whose power of some order belong to $\mathfrak{I}$

$$r(\mathfrak{I}) = \{f \in k[z_1, ..., z_n] : \exists n \geq 0, f^n \in \mathfrak{I}\}.$$

Exercise 1.14. *Show that $r(\mathfrak{I})$ is an ideal containing $\mathfrak{I}$.*

The following theorem of D. Hilbert is fundamental for Algebraic Geometry.

Theorem 1.15 (Hilbert's Nullstellensatz)**.** *Let $\mathfrak{I} \subset k[z_1, ..., z_n]$ be an ideal. Let $V(\mathfrak{I})$ be the solution set (also called "zero set") of $\mathfrak{I}$. Then*

$$I(V(\mathfrak{I})) = r(\mathfrak{I}).$$

NOTE: Nullstellen = zero locus - the loci where a function vanishes.

Example 1.16. If $\mathfrak{p} \subset k[z_1, ..., z_n]$ is a prime ideal, then, by definition,

$$\mathfrak{p} = r(\mathfrak{p}).$$

Thus, if $Z = V(\mathfrak{p})$ - the zero locus of $\mathfrak{p}$, then

$$\mathfrak{p} = I(Z).$$

We definite a topology on $\mathbb{A}_k^n$ by specifying algebraic sets as closed sets of the topology. Hence, an open set has the form

$$D_{\mathfrak{I}} = \{P \in \mathbb{A}_k^n : \exists f \in \mathfrak{I}, f(P) \neq 0\}$$

for a given ideal $\mathfrak{I}$ of $k[z_1, ..., z_n]$.

In particular, for a singe polynomial $f \in k[z_1, ..., z_n]$,

$$D_f = \{P \in \mathbb{A}_k^n : f(P) \neq 0\}.$$

These open sets are called basic open sets, they form a basis for the Zarisky topology on $\mathbb{A}_k^n$.

Exercise 1.17. *Check that the above definition is correct, i.e., show that the above defined open set form a topology.*

Example 1.18. In the case $n = 1$, the affine line $\mathbb{A}_k^1$. Every ideal if $k[z]$ is principal, i.e., generated by a singe polynomial. Hence, closed sets in $\mathbb{A}_k^1$ are either the whole space or finite sets (and the empty set). We thus see that this topology is very coarse (i.e., there are very few open sets). Nevertheless, it is quite useful. It is called Zariski topology.

Noetherianity. The ring $k[z_1, ..., z_n]$ is Noetherian, i.e., any increasing chain of ideals

$$\mathfrak{I}_1 \subset \mathfrak{I}_2 \subset ... \subset \mathfrak{I}_r \subset ...$$

stabilizes. This is reflexed on the Zariski topology: any decreasing chain of closed subsets in $\mathbb{A}_k^n$ stabilizes. This implies the existence of irreducible components in any closed subset.

A closed subset Z is called irreducible if it cannot be represented as union of two proper closed subsets.

Example 1.19. If $\mathfrak{p}$ is a prime ideal then $V(\mathfrak{p})$ is irreducible.

Exercise 1.20. *Show that a closed setin $\mathbb{A}_k^n$ is the union of finitely many irreducible closed subsets.*

Thus we see that irreducible closed subsets are cornerstones in the Zariski topology.

Definition 1.21. An irreducible algebraic set in $\mathbb{A}_k^n$ is called an affine variety. It has a topology inherited from the Zariski topology on $\mathbb{A}_k^n$.

Hence, a closed subset in an affine variety $Z \subset \mathbb{A}_k^n$ has the form

$$V(\mathfrak{I}) \cap Z,$$

where $\mathfrak{I} \subset k[z_1, ..., z_n]$.

Recall that if $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$, then

$$V(\mathfrak{I}) \cap V(\mathfrak{p}) = V(\mathfrak{I} + \mathfrak{p}).$$

In this way we obtain a 1–1 correspondence between closed subsets in Z and radical ideals in $k[z_1, ..., z_n]$ containing $\mathfrak{p}$

$$\{Z' \subseteq Z, \text{closed subsets}\} \longleftrightarrow \{I \supseteq \mathfrak{p}, \text{ radial ideals}\}.$$

Let

$$A(Z) := k[z_1, ..., z_n]/\mathfrak{p},$$

and

$$\varphi : k[z_1, .., z_n] \longrightarrow A$$

be the canonical quotient map. Then we have the correspondence

$$\{I \supseteq \mathfrak{p}, \text{ radial ideals}\} \longleftrightarrow \{I \text{ radial ideals of } A\}.$$

In this way, closed subsets of Z are in 1–1 correspondence with radical ideals in $A(Z)$.

In particular, a point of Z corresponds to a maximal ideal of $k[z_1, ..., z_n]$ containing $\mathfrak{p}$, and hence, a maximal ideal of A.

We remark that, by definition, A is an integral domain: if $a.b = 0$ the either $a = 0$ or $b = 0$.

We call A the affine coordinate ring of Z. Notice that we can fully recover Z from A:

- Points $\longleftrightarrow$ maximal ideals;
- Closed subsets $\longleftrightarrow$ radical ideals.

Thus an affine variety is completely determined by its coordinate ring. This is independent of the ambient affine space $\mathbb{A}_k^n$.

The homogeneous coordinate ring of a projective variety. There are analogous definitions for projective varieties. Let $X \subset \mathbb{P}_k^n$ be a closed subset. Thus, the homogeneous coordinate of points in X is the solution set of a homogeneous ideal $\mathfrak{J}$ in $k[Z_0, Z_1, ..., Z_n]$. Let $I(X)$ be the ideal of polynomials vanishing on X. Then,

$$\mathfrak{J} \subset I(X).$$

As in the affine case $I(X)$ is the radical of $\mathfrak{J}$ and we have a 1–1 correspondence between the set of closed subsets in $\mathbb{P}_k^n$ and homogeneous ideals of $k[Z_0, Z_1, ..., Z_n]$ OTHER THAN the ideal $(Z_0, Z_1, ..., Z_n)$ (since this ideal corresponds to the point $(0, 0, ..., 0) \in \mathbb{A}_k^{n+1}$).

An algebraic set X in $\mathbb{P}^n_k$ is irreducible if the ideal $I(X)$ is a prime ideal. We define the homogeneous coordinate ring of X to be

$$S(X) := k[Z_0, Z_1, ..., Z_n]/I(X).$$

This is a graded ring because $I(X)$ is a homogeneous ideal.

RECALL: A ring R is graded if it has the form

$$R = R_0 \oplus R_1 \oplus ... \oplus R_n \oplus ...$$

such that, if $x \in R_m, y \in R_n$ then $xy \in R_{m+n}$. In particular, R_0 is a subring in R and each R_i is a module over R_0.

For example, $R = k[Z_0, Z_1, ..., Z_n]$ is graded by the total degree of the homogeneous polynomials, for each $f \in R$,

$$f = f_0 + f_1 + ... + f_d,$$

where f_i is the homogeneous component of degree i. Hence,

$$R = R_0 \oplus R_1 \oplus ... \oplus R_n \oplus ...$$

By definition, a homogeneous ideal $\mathfrak{J}$ in R is the direct sum of its homogeneous components:

$$\mathfrak{J} = \mathfrak{J}_0 \oplus \mathfrak{J}_1 \oplus ... \oplus \mathfrak{J}_n \oplus ...; \mathfrak{J}_i \subset R_i.$$

Hence,

$$R/\mathfrak{J} = R_0/\mathfrak{J}_0 \oplus R_1/\mathfrak{J}_1 \oplus ... \oplus R_n/\mathfrak{J}_n \oplus ...$$

Exercise 1.22. *Show that the above decomposition make $R/\mathfrak{J}$ a graded ring.*

WARNING: The construction of the homogeneous coordinate ring of a projective variety is dependent on the coordinate, i.e., it is dependent on the ambient projective space.

Example 1.23. Consider again the Fermat equation

$$Z_0^2 + Z_1^2 = Z_2^2$$

in the ring $k[Z_0, Z_1, Z_2]$. It defines a variety X (a curve in $\mathbb{P}_k^2$). The coordinate ring $S(X) = k[Z_0, Z_1, Z_2]/(Z_0^2 + Z_1^2 - Z_2^2)$ has:

- $S_0 = k$;
- $S_1 = \langle Z_0, Z_1, Z_2 \rangle_k$ is a 3-dimensional $k-$vector space;
- $S_2 = \langle Z_0^2, Z_1^2, Z_0Z_1, Z_0Z_2, Z_1Z_2 \rangle_k$ is a $5-$dimensional $k-$vector space;
- ...

Consider now the map from $\mathbb{P}_k^1$ to $\mathbb{P}_k^2$ given by

$$[U : V] \longmapsto [U^2 - V^2 : 2UV : U^2 + V^2].$$

Its image is in X.

Conversely, consider the map

$$[Z_0 : Z_1 : Z_2] \longmapsto [Z_0 + Z_2 : Z_1],$$

restricted to X. This map is well-defined up to the point $[1 : 0 : -1]$. To prolong it to this point, we perform the following transformation:

$$\frac{Z_0 + Z_2}{Z_1} = \frac{Z_2^2 - Z_0^2}{(Z_2 - Z_0)Z_1} = \frac{Z_1}{Z_2 - Z_0}.$$

Thus X is isomorphic to the projective line.

But the homogeneous coordinate ring $S(X)$ is **NOT** isomorphic to $k[U, V]$ as graded rings. We see that the algebra-geometric properties of projective varieties is more complicated that that of affine varieties (and much more beautiful).

Remark 1.24. The isomorphism considered above explains why the Fermat equation $x^2+y^2 = z^2$ is solvable in in the integers. For higher degree equations, such morphisms do not exist.

2. Regular and rational functions and maps

2.1. Regular functions. Let $X \subset \mathbb{A}^n_k$ be a closed subset. Any polynomial in $k[z_1, ..., z_n]$ determines a function of X. Any two such functions are equal if the polynomials are congruent modulo $I(X)$, that is,

$$f \equiv g \pmod{I(X)} \Longrightarrow f = g \quad \text{as functions on } X.$$

Such a function is called a regular function on X. In particular, if X is an affine variety, regular functions on X are in 1-1 correspondence with elements of the coordinate ring of X.

The above definition does not reflect the "local feature of function" - a function is regular or not is decided by the whole space X. Next, we shall give a local notion of regularity of a function and show that the two definitions are equivalent.

Let $P \in \mathbb{A}^n_k$ be a point. Let f be a function on an open subset $U \subset X$ containing P. We say that f is regular at P if in some neighborhood V of P it is can be expressed as a quotient

$$f = \frac{g}{h},$$

where g, h are polynomials with $h \neq 0$ on V.

Notice that the choice of the neighborhood of P is arbitrary. We emphasize that open sets in $\mathbb{A}_k^n$ is very big!

Assume that f is defined over an open subset U. We say that f is regular on U if f is regular at every point of U.

Example 2.1 (Every regular function on $\mathbb{A}_k^n$ is polynomial)**.** Let f be a function of $\mathbb{A}_k^n$, which is regular at every point of $\mathbb{A}_k^n$. Thus, for each point P, we have

$$f = \frac{g_P}{h_P},$$

where g_P, h_P are polynomials with $h_P(P) \neq 0$.

Consider the ideal generated by all polynomials $\{h_P | P \in \mathbb{A}_k^n\}$. Then this ideal does not have common zero (as f is defined every where)! Hence, according to Hilbert's Nullstellensatz, it is the whole polynomial ring and thus 1 is in it. Therefore,

$$1 = \sum c_P \cdot h_P \text{ - a finite sum with } c_P \in k[z_1, ..., z_n].$$

This implies that

$$f = \frac{g_P}{h_P} = \frac{\sum c_P g_P}{\sum c_P h_P} = \sum c_P g_P \in k[z_1, ..., z_n].$$

This shows that f is a polynomial.

Let $Z \subset \mathbb{A}_k^n$ be a closed subset, $P \in Z$. Let U be a neighborhood of P in Z and f is a function on U. We say that f is regular at P if in some neighborhood V of P, f is expressible as a quotient $\dfrac{p}{q}$, where p, q are polynomials in $k[z_1, ..., z_n]$ and $q \neq 0$ on V.

Note that polynomials that are congruent modulo $I(Z)$ determine the same function on Z. Hence, we can consider p and q as elements of $A(Z) = k[z_1, ..., z_n]/I(Z)$ the coordinate ring of Z. The same argument as above shows

Lemma 2.2. *The ring of functions regular at every point of Z is the coordinate ring $A(Z)$ of Z. More generally, the ring of functions regular at every point of a basic open set D_f is the localization $A(Z)[\frac{1}{f}]$.*

The lemma above explains the meaning of localization in Commutative Algebra: roughly speaking, localization is to restrict to an open set.

Exercise 2.3. *What is the ring of regular functions S on the open $\mathbb{A}_k^2 \setminus \{(0,0)\}$. (Note that this open set is NOT a basic open subset).*

HINT: Let represent this open subset as the union of two basic open subsets: $\mathbb{A}^2_k \setminus (Ox)$ and $\mathbb{A}^2_k \setminus (Oy)$.

Last but not least, we can speak about functions regular at a point P in an affine variety Z without specifying the open set on which it is defined (but it has to be defined in some open set containing P).

Given any two such functions, say f_1 and f_2, defined respectively on open sets V_1 and V_2 containing P. Then $V_1 \cap V_2$ is also open set and contains P. Hence, we can consider f_1 and f_2 as a function on $V_1 \cap V_2$ and perform addition and multiplication of them on this open set.

Now we say f_1 and f_2 are "the same" at P if there exists an open neighborhood on which they are equal. In this case, we also say that f_1 and f_2 determine the same "germ of functions" at P. A germ of regular functions at P is thus determined by a function on an open neighborhood of P, regular at P. As shown above, we can add multiply germs of regular function. In this way we obtain the ring of germs of functions regular at P.

Lemma 2.4. *The ring of germs of function regular at P is isomorphic to the localization of $A(Z)$ at the maximal ideal of $A(Z)$ determining P, i.e.,*

$$\mathfrak{m}_P = \{f \in A(Z) | f(P) = 0\}.$$

More explicitly, denote the ring of germs of function regular at P by $\mathcal{O}_{Z,P}$, we have

$$\mathcal{O}_{Z,P} = \{\frac{p}{q} \quad | \quad p, q \in A(Z), q(P) \neq 0\}.$$

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