Automorphism groups of projective varieties

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Introduction

The objects of the talk are the *automorphisms* of a projective variety X over an algebraically closed field k, i.e., the morphisms of varieties $f : X \to X$ such that there exists a morphism of varieties $g : X \to X$ satisfying $f \circ g = g \circ f = \mathrm{id}$. For simplicity, we will assume k of characteristic 0.

The automorphisms form an (abstract) group $\mathrm{Aut}(X)$, which is only partially understood. We will first see that $\mathrm{Aut}(X)$ has the structure of a k-group scheme which is *locally of finite type*, i.e., a union of open affine subschemes of finite type.

One then shows that the connected component of the identity in $\mathrm{Aut}(X)$ is a closed normal subgroup scheme of finite type. By Cartier's theorem, this subgroup scheme $\mathrm{Aut}^0(X)$ is a smooth variety.

Moreover, the connected components of $\mathrm{Aut}(X)$ are the cosets $f\,\mathrm{Aut}^0(X)$, where $f \in \mathrm{Aut}(X)$. They are parameterized by the quotient group $\mathrm{Aut}(X)/\mathrm{Aut}^0(X) = \pi_0\,\mathrm{Aut}(X)$.

Introduction (continued)

Thus, we have an exact sequence

$$1 \longrightarrow \mathrm{Aut}^0(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \pi_0 \mathrm{Aut}(X) \longrightarrow 1,$$

where $\mathrm{Aut}^0(X)$ is a connected *algebraic group* (i.e., a group scheme of finite type), and $\pi_0 \mathrm{Aut}(X)$ is a discrete group.

We may analyze $\mathrm{Aut}(X)$ by considering its *neutral component* $\mathrm{Aut}^0(X)$ and its *group of components* $\pi_0 \mathrm{Aut}(X)$ separately. In this lecture, we will mainly consider the group of components. The neutral component will be discussed in my lecture at the Workshop on Commutative Algebra and Algebraic Geometry.

Examples show that $\pi_0 \mathrm{Aut}(X)$ may be infinite; equivalently, $\mathrm{Aut}(X)$ is not necessarily of finite type. Further examples (more elaborate and very recent) show that the group $\pi_0 \mathrm{Aut}(X)$ may not be finitely generated.

Still, this group admits a homomorphism with finite kernel to some $\mathrm{GL}_r(\mathbb{Z})$, and this has remarkable finiteness consequences.

The automorphism functor

In this first part, we consider varieties and schemes over an algebraically closed field k. Morphisms and products of schemes are understood to be over k as well.

Definition

Let X be a scheme. A *family of automorphisms of X over a scheme S* is an automorphism of the S-scheme $X \times S$, i.e., an automorphism f of $X \times S$ such that $\mathrm{pr}_S \circ f = \mathrm{pr}_S$, where $\mathrm{pr}_S : X \times S \to S$ denotes the projection.

For example, a family of automorphisms of X over Spec(k) is just an automorphism of X.

More generally, a family of automorphisms of X over S is a morphism of schemes $f : X \times S \to X \times S$ of the form $(x, s) \longmapsto (F(x, s), s)$, where $F : X \times S \to X$ is a morphism, such that f admits an inverse of the same form. In particular, the morphism $F_s : X \longrightarrow X$, $x \longmapsto F(x, s)$ is an automorphism for any $s \in S(k)$ (but the condition for F to define a family of automorphisms is more restrictive).

The families of automorphisms of X over S form a group denoted $\mathrm{Aut}(X \times S/S)$.

Definition

Let $f \in \mathrm{Aut}(X \times S/S)$, and $u : T \to S$ a morphism of schemes. The *pull-back* $u^*(f)$ is the automorphism of $X \times T$ over T obtained by base change: $u^*(f)(x,t) = f(x, u(t))$.

This defines a group homomorphism $u^* : \mathrm{Aut}(X \times S/S) \to \mathrm{Aut}(X \times T/T)$.

The assignments $S \mapsto \mathrm{Aut}(X \times S/S)$ and $u \mapsto u^*$ yield a contravariant functor from the category of schemes to that of groups: the *automorphism group functor* $\mathbf{Aut}_X$.

Theorem

If X is a projective scheme, then $\mathbf{Aut}_X$ is represented by a group scheme, locally of finite type.

The idea of the proof is to encode automorphisms by their graphs, which are closed subschemes of $X \times X$, and to use the representability of the *Hilbert functor* of such subschemes.

More specifically, one assigns to each $f \in \mathrm{Aut}(X)$ the morphism

$$(\mathrm{id}, f) : X \longrightarrow X \times X, \quad x \longmapsto (x, f(x)).$$

This morphism yields an isomorphism of X onto its image, the *graph* Γ_f. This is the inverse image of the diagonal $\mathrm{diag}(X) \subset X \times X$ under the morphism $X \times X \longrightarrow X \times X$, $(x, y) \longmapsto (f(x), y)$, and hence is closed.

The two projections $\mathrm{pr}_1, \mathrm{pr}_2 : X \times X \to X$ restrict to isomorphisms $\Gamma_f \xrightarrow{\simeq} X$, and this condition characterizes the graphs of automorphisms among the closed subschemes of $X \times X$. The graph of the identity is the diagonal $\mathrm{diag}(X)$.

The graph construction

$$f \in \mathrm{Aut}(X) \longmapsto \Gamma_f \subset X \times X$$

extends to families of automorphisms over a scheme S. It identifies these families with the closed subschemes

$$\Gamma \subset (X \times S) \times_S (X \times S) \simeq X \times X \times S$$

such that the two projections $\mathrm{pr}_i : \Gamma \longrightarrow X \times S, (x_1, x_2, s) \longmapsto (x_i, s)$ are isomorphisms.

Some background on flatness

Definition

Let A be a commutative ring. An A-module M is *flat* if for any exact sequence of A-modules $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$, the sequence of A-modules $0 \longrightarrow M \otimes_A M_1 \longrightarrow M \otimes_A M_2 \longrightarrow M \otimes_A M_3 \longrightarrow 0$ is exact.

We will need the following observations:

1) If M is a flat A-module and B is an A-algebra, then the B-module $B \otimes_A M$ is flat.

2) Every locally free A-module is flat. In particular, if A is a field, then every A-module is flat.

There is a partial converse to 2): every finitely generated flat module over a noetherian ring is locally free.

Definition

A morphism of schemes $f : X \to Y$ is *flat* if for any $x \in X$ with image $y = f(x)$, the homomorphism of local rings $f^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ a flat module over $\mathcal{O}_{Y,y}$. We also say that X *is flat over* Y.

Flat morphisms and the Hilbert polynomial

Flatness is preserved by base change: if $f : X \to Y$ is a flat morphism of schemes and $Y' \to Y$ is any morphism of schemes, then the induced morphism $f' : X \times_Y Y' \to Y'$ is flat.
Also, the structural morphism $f : X \to \mathrm{Spec}(k)$ is flat for any scheme X. As a consequence, the projection $X \times Y \to Y$ is flat for any schemes X, Y.

There is an important flatness criterion in terms of the *Hilbert polynomial*. Recall that the Hilbert polynomial of a closed subscheme $X \subset \mathbb{P}^n$ is the unique polynomial $P(z) \in \mathbb{Q}[z]$ such that $P(\ell) = \dim H^0(X, \mathcal{O}(\ell))$ for $\ell \gg 0$. This is also the Hilbert polynomial of the homogeneous coordinate ring of X. The leading term of $P(z)$ is $\frac{d}{r!}z^r$, where r is the dimension of X, and d is its degree.

Proposition

Let S be a connected noetherian scheme. Let X be a closed subscheme of $\mathbb{P}^n_S = \mathbb{P}^n \times S$ with projection $f : X \to S$. For any $s \in S$, consider the Hilbert polynomial P_s of the fiber X_s viewed as a closed subscheme of $\mathbb{P}^n_{k(s)}$. If f is flat, then P_s is independent of $s \in S$. The converse holds when S is reduced.

The Hilbert functor

See Hartshorne, Algebraic Geometry, Chapter III for more on flatness and the Hilbert polynomial.

Definition

Let $P(z) \in \mathbb{Q}[z]$. A *flat family of closed subschemes of* $\mathbb{P}^n$ *with Hilbert polynomial* P *over a scheme* S is a closed subscheme of $\mathbb{P}^n_S$ which is flat over S and satisfies $P_s = P$ for any $s \in S$.

For any such subscheme X, the fibers X_s have the same dimension and degree by the above proposition.

Given a morphism of schemes $u : T \to S$, the pull-back $X \times_S T$ is a closed subscheme of $\mathbb{P}^n_T$ which is flat over T, with Hilbert polynomial P. This yields the *Hilbert functor* $\mathbf{Hilb}^P_{\mathbb{P}^n}$.

The following fundamental existence result is due to Grothendieck in a much greater generality.

Theorem

The functor $\mathbf{Hilb}^P_{\mathbb{P}^n}$ *is represented by a projective scheme.*

The Hilbert scheme

We will need a slight generalization of Grothendieck's existence result, in which $\mathbb{P}^n$ is replaced with a projective scheme Y.

We define the Hilbert functor $\mathbf{Hilb}_Y$ in the obvious way, by considering flat families of closed subschemes of Y instead of $\mathbb{P}^n$ (but we do not specify the Hilbert polynomial).

Theorem

The functor $\mathbf{Hilb}_Y$ *is represented by a scheme* Hilb_Y*, the disjoint union of open and closed projective schemes.*

This follows from Grothendieck's theorem by embedding Y in some $\mathbb{P}^n$ and showing that "X is a closed subscheme of Y" is a closed condition.

In particular, Hilb_Y is locally of finite type. But it is not of finite type if $\dim(Y) > 0$, since Hilb_Y contains disjoint non-empty open subschemes U_m ($m \geq 1$) parameterizing the finite subsets of $Y(k)$ with m elements.
In fact, each U_m is open in the Hilbert scheme Hilb_Y^m representing flat families of finite subschemes of length m of Y; the corresponding Hilbert polynomial is $P(t) = m$.

Representability of the automorphism functor

We return to the automorphism functor $\mathbf{Aut}_X$, where X is a projective scheme. For any scheme S, the graph construction identifies $\mathbf{Aut}_X(S)$ with the set of closed subschemes $\Gamma \subset X \times X \times S$ such that the two projections $\Gamma \to X \times S$ are isomorphisms. Thus, the projection $\Gamma \to S$ is obtained from either projection $X \times S \to S$ via an isomorphism over S. So Γ is flat over S.

This identifies $\mathbf{Aut}_X$ with a subfunctor of the Hilbert functor $\mathbf{Hilb}_{X \times X}$. Next, one shows that $\mathbf{Aut}_X$ is represented by an open subscheme Aut_X of $\mathrm{Hilb}_{X \times X}$. For this, denoting by $\mathrm{pr}_1, \mathrm{pr}_2 : X \times X \to X$ the projections, one shows that "pr_i is an isomorphism" is an open condition. Since $\mathrm{Hilb}_{X \times X}$ is locally of finite type, its open subscheme Aut_X is locally of finite type, too.

In particular, the neutral component Aut_X^0 is open in the connected component of the diagonal in $\mathrm{Hilb}_{X \times X}$, which is a projective scheme. So the closure of Aut_X^0 in $\mathrm{Hilb}_{X \times X}$ is a projective variety.

If $X = \mathbb{P}^n$ then Aut_X is the projective linear group PGL_{n+1} (the quotient of the general linear group GL_{n+1} by its center, the scalar matrices). This is a connected affine algebraic group. Its closure in $\mathrm{Hilb}_{X \times X}$ is the *variety of complete collineations* (M. Thaddeus, Math. Ann. 1999).

Examples of automorphism groups

We now assume that the field k is (algebraically closed) of characteristic 0. Then Aut_X is a smooth variety and hence we may identify it with $\mathrm{Aut}(X)$.

1) Let X be a *finite scheme*, i.e., $X = \mathrm{Spec}(A)$ where A is a k-algebra of finite dimension as a k-vector space. Then $\mathrm{Aut}(X)$ is the closed subgroup scheme of the general linear group $\mathrm{GL}(A)$ consisting of those linear maps f such that $f(ab) = f(a)f(b)$ for all $a, b \in A$. In particular, $\mathrm{Aut}(X)$ is affine.

2) Let X be an *elliptic curve*, i.e., a smooth projective curve of genus 1. Choosing a k-rational point $0 \in X$ defines a commutative algebraic group law $+$ on X with neutral element 0. In particular, for any $x \in X$, we have the translation $\tau_x : X \longrightarrow X$, $y \longmapsto x + y$. This yields a homomorphism $\tau : X \to \mathrm{Aut}(X)$. Thus, every $f \in \mathrm{Aut}(X)$ can be written uniquely as $\tau_x \circ g$, where $x \in X$ and $g \in \mathrm{Aut}(X)$ satisfies $g(0) = 0$. Then g is a homomorphism, and hence $g\tau_x g^{-1} = \tau_{g(x)}$. It follows that $\mathrm{Aut}(X) \simeq X \rtimes \mathrm{Aut}(X, 0)$ via $(x, g) \mapsto \tau_x \circ g$. Also, $\mathrm{Aut}(X, 0)$ is a finite group of order 2, 4 or 6.

In particular, $\mathrm{Aut}^0(X) \simeq X$ is not affine. Moreover, $\pi_0 \, \mathrm{Aut}(X) \simeq \mathrm{Aut}(X, 0)$ is non-trivial, and hence $\mathrm{Aut}(X)$ is a non-connected algebraic group.

3) Let X be a smooth projective curve of genus g.
If $g = 0$ then $X \simeq \mathbb{P}^1$ and $\mathrm{Aut}(X) \simeq \mathrm{PGL}_2$ acting via
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] = [ax + by : cx + dy].$$
If $g = 1$ then X is an elliptic curve (see Example 2).

If $g \geq 2$ then $\mathrm{Aut}(X)$ is a finite group of order $\leq 84(g-1)$. Moreover, every finite group can be obtained in this way (Hurwitz, Math. Ann. 1893).

4) Let $X = Y \times Y$, where Y is an elliptic curve with origin 0. Then X is a commutative algebraic group with neutral element $(0,0)$.

One shows as above that $\mathrm{Aut}(X) \simeq X \rtimes \mathrm{Aut}(X,(0,0))$, where $\mathrm{Aut}(X,(0,0))$ is a discrete group. Thus, we have again $\mathrm{Aut}^0(X) \simeq X$ and $\pi_0 \mathrm{Aut}(X) \simeq \mathrm{Aut}(X,(0,0))$.

But now $\mathrm{Aut}(X,(0,0))$ is infinite, as it contains the group $\mathbb{Z}$ acting via $n \cdot (y_1, y_2) = (y_1 + ny_2, y_2)$. Thus, $\mathrm{Aut}(X)$ is not of finite type.

In fact, $\pi_0 \mathrm{Aut}(X) \supset \mathrm{GL}_2(\mathbb{Z})$ acting via linear combinations of y_1, y_2. Moreover, equality holds if and only if $\mathrm{End}(Y) = \mathbb{Z}$, i.e., Y has no complex multiplication.

5) Let X be an *abelian variety*, i.e., an algebraic group which is a projective variety (for instance, an elliptic curve as in Example 3, or a product of such curves as in Example 4). Then the group law of X is commutative and one obtains as in these examples $\mathrm{Aut}^0(X) \simeq X$, $\pi_0 \mathrm{Aut}(X) \simeq \mathrm{Aut}(X, 0)$.

Moreover, the group $\mathrm{Aut}(X, 0)$ is finitely generated by a result of Borel and Harish-Chandra (Ann. Math. 1962).

6) It has been a folklore conjecture that the group $\pi_0 \mathrm{Aut}(X)$ is finitely generated for any smooth projective variety X. But this was disproved by Lesieutre via a counterexample in dimension 6 (Invent. Math. 2018)

Then Tien-Cuong Dinh and Keiji Oguiso constructed a smooth projective *surface* X over $\mathbb{C}$ such that $\mathrm{Aut}(X)$ is discrete and not finitely generated (Adv. Math. 2020).

Also, for any integer $n \geq 3$, there exists a smooth projective *rational* variety X of dimension n over $\mathbb{C}$, such that $\mathrm{Aut}(X)$ is discrete and not finitely generated (Dinh, Oguiso and Xun Yu, arXiv:2002.04737).

It is unknown whether there exists a smooth projective rational surface with this property.

Two general results

We consider a projective scheme X over an (algebraically closed) field k of characteristic 0.

Proposition

If X is a curve, then $\mathrm{Aut}(X)$ *is an algebraic group.*

Equivalently, $\pi_0 \mathrm{Aut}(X)$ is finite. To prove this, one shows that the graphs of automorphisms have only finitely many Hilbert polynomials, by using the Riemann-Roch theorem.

Theorem

There exists a homomorphism $\rho : \pi_0 \mathrm{Aut}(X) \longrightarrow \mathrm{GL}_r(\mathbb{Z})$ *with finite kernel.*

The homomorphism ρ is constructed via the action of automorphisms on divisors. More specifically, the numerical equivalence classes of Cartier divisors on X form a free abelian group $\mathrm{N}^1(X)$ of finite rank: the *Picard number* of X. Moreover, the natural action of $\mathrm{Aut}(X)$ on $\mathrm{N}^1(X)$ factors through an action of $\pi_0 \mathrm{Aut}(X)$. The class of a hyperplane has a finite stabilizer for this action.

Corollary

There exists a positive integer $N = N(X)$ such that every finite subgroup of $\pi_0 \operatorname{Aut}(X)$ has order at most N.

Indeed, this holds for $\mathrm{GL}_r(\mathbb{Z})$ (Minkowski, J. Crelle, 1887). It would be very interesting to express $N(X)$ in terms of numerical invariants of X. This is known for special classes of projective varieties.

Corollary

The intersection of all subgroups of finite index of $\pi_0 \operatorname{Aut}(X)$ is finite.

Indeed, the intersection of all subgroups of finite index of $\mathrm{GL}_r(\mathbb{Z})$ is trivial: consider the kernels of the "reduction mod N" homomomorphisms $\mathrm{GL}_r(\mathbb{Z}) \to \mathrm{GL}_r(\mathbb{Z}/N\mathbb{Z})$.

It is an open question whether $\pi_0 \operatorname{Aut}(X)$ is *residually finite*, i.e., the intersection of its subgroups of finite index is trivial.

One may also ask whether $\pi_0 \operatorname{Aut}(X)$ is *topologically finitely generated*, i.e., there are finitely many automorphisms $f_1, \ldots, f_m$ which generate every finite quotient group.