Semicontinuity of the solution map to a parametric optimal control problem

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Abstract. This paper studies the solution stability of a parametric optimal control problem governed by nonlinear ordinary differential equations and nonconvex cost functions with control constraints. By using the direct method, the Pontryagin principle and exploiting structures of the problem, we obtain upper semicontinuity and continuity of the solution map with respect to parameters.

Key words: Parametric optimal control problem, existence of solution, solution stability, lower semicontinuity, upper semicontinuity, continuity.

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1 Introduction

In this paper we study the following parametric optimal control problem. Determine a control vector $u \in L^p([0,1],\mathbb{R}^m)$ with $1 < p < \infty$ and a trajectory $x \in W^{1,1}([0,1],\mathbb{R}^n)$ which minimize the cost function

$$J(x,u,\mu) := \int_0^1 f(t,x(t),u(t),\mu(t))dt \tag{1}$$

with the state equation

$$\begin{cases} \dot{x}(t) = A(t,x(t)) + B(t,x(t))u(t) + T(t,\lambda(t)) \text{ a.e. } t \in [0,1], \\ x(0) = x_0 \end{cases} \tag{2}$$

and the control constraint

$$a(t) \le u(t) \le b(t) \quad \text{a.e.} \quad t \in [0,1]. \tag{3}$$

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Here (μ, λ) is a couple of parameters which belongs to $L^r([0,1], \mathbb{R}^k) \times L^s([0,1], \mathbb{R}^l)$ with $1 \le r, s \le \infty$, $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ is a function, $A(t,x)$ is an $n \times 1$ matrix, $B(t,x)$ is an $n \times m$ matrix and $T(t, \lambda)$ is an $n \times 1$ matrix and $a, b \in L^p([0,1], \mathbb{R}^m)$. Note that constraint (3) means $a_i(t) \le u_i(t) \le b_i(t)$ for $i = 1, 2, ..., m$.

Recall that $W^{1,1}([0,1], \mathbb{R}^n)$ is a Sobolev space which consists of absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$ such that $\dot{x} \in L^1([0,1], \mathbb{R}^n)$ and $C([0,1], \mathbb{R}^n)$ is a Banach space of continuous vector functions $y : [0,1] \to \mathbb{R}^n$. Their norms are given by

$$\|x\|_{1,1} = |x(0)| + \|\dot{x}\|_1, \ \|y\|_0 = \sup_{t \in [0,1]} |y(t)|,$$

respectively. Let us put

$$X = W^{1,1}([0,1], \mathbb{R}^n), U = L^p([0,1], \mathbb{R}^m), Z = X \times U,$$
$$M = L^r([0,1], \mathbb{R}^k), \ \Lambda = L^s([0,1], \mathbb{R}^l)$$

and define $K(\lambda)$ for $\lambda \in \Lambda$ by setting

$$K(\lambda) = \{z = (x,u) \in X \times U | (2) \text{ and } (3) \text{ are satisfied}\}. \tag{4}$$

Then $(1) - (3)$ can be reformulated in the form

$$P(\mu, \lambda) \qquad \begin{cases} J(z, \mu) \to \inf, \\ z \in K(\lambda). \end{cases} \tag{5}$$

Throughout this paper we denote by $S(\mu, \lambda)$ the solution set of $(1) - (3)$ or $P(\mu, \lambda)$ corresponding to parameter (μ, λ). We denote by $(\overline{\mu}, \overline{\lambda})$ the reference point and call $P(\overline{\mu}, \overline{\lambda})$ the unperturbed problem.

Our main concern is to investigate the behavior of $S(\mu, \lambda)$ when (μ, λ) varies around $(\overline{\mu}, \overline{\lambda})$. This problem has been interesting to several authors in the last decade. For papers which have a close connection to the present work, we refer the readers to [8], [9], [15]-[19] and the references given therein.

It is known that when $J(\cdot, \cdot, \mu)$ is strongly convex for all μ and $K(\lambda)$ is a convex set, then the solution map of (5) is single-valued. In this case, under certain conditions, Dontchev [9] showed that the solution map is continuous with respect to parameters.

Recently Malanowski [15]-[19] showed that if weak second-order optimality conditions and standard constraints qualifications are satisfied at the reference point, then the solution map is a Lipschitz continuous function of parameters. The obtained results in [15]-[19] were proved by techniques of implicit function theorem. Note that the obtained results in [15]-[19] are of problems subject to state constraints without control constraints.

When conditions mentioned above are invalid, the solution map may not be singleton. In this situation, we have to use tools of set-valued analysis to treat the problem. Such a treatment has been developed recently by Kien et al. [12] and [13]. In [12] and [13] the authors studied the lower semicontinuous property of the solution map to problem (1) − (3) in the case where the state equation is linear and the cost function is convex in both variables. For this case, the authors showed that if the unperturbed problem is good enough, then the solution map is lower semicontinuous at the reference point.

In this paper we continue to develop the method in [12] and [13] in order to study the upper semicontinuity and continuity of the solution map $S(\mu, \lambda)$ of problem (1) − (3). Here in problem (1) − (3), the state equation is nonlinear and the cost function is not required to be convex in both variables.

It is noted that in the case of finite-dimension spaces, the upper semicontinuity of the solution map to parametric mathematical programming problems is easy to obtain. The reason is that the upper semicontinuity of S is equivalent to the closeness of its graph. It is well known that if S has a closed graph and uniformly compact, that is, there exists a compact set D in the strong topology such that $S(\mu, \lambda) \subset D$ for all (μ, λ) in a neighborhood of $(\overline{\mu}, \overline{\lambda})$ then S is upper semicontinuous at $(\overline{\mu}, \overline{\lambda})$ (see [4, Corollary, p.112] and [11, Theorem 3.1]). Unfortunately, in the infinite-dimensional setting of problem (1)–(3), although each set $S(\mu, \lambda)$ is a weakly compact set, the family $\{S(\mu, \lambda)\}$ is not strongly uniformly compact. Hence, the closeness of graph of S is far from the upper semicontinuity of S.

In our paper, by using the direct method, the Pontryagin Maximum Principle and exploiting structures of the problem, we show that under certain conditions, the solution map is $(s, w)-$upper semicontinuous at reference point (see Definition 2.1 for $(s, w)-$upper semicontinuity). Besides, we also show that if the unperturbed problem is good enough, then the solution map is $(s, s)-$continuous with respect to parameters at the reference point. It is worth pointing out that our proofs are based on the direct method and analyzing first order optimality conditions (Pontryagin's Principle) of the problem. We do not use second-order optimality conditions for the proof as usual.

The paper is organized as follows. In Section 2, we recall some notions of set-valued analysis and state our main results. Section 3 is destined for some auxiliary results. The proofs of main results are given in Section 4.

2 Statement of main result

Let us assume that $F : E_1 \rightrightarrows E_2$ is a multifunction between topological spaces. We denote by $\text{dom}F$ and $\text{gph}F$ the effective domain and the graph of F, respectively, where

$$\text{dom}F := \{z \in E_1 | F(z) \neq \emptyset\}$$

and

$$\text{gph}F := \{(z, v) \in E_1 \times E_2 | v \in F(z)\}.$$

A multifunction F is said to be *lower semicontinuous* at $z_0 \in E_1$ if for any open set V_0 in E_2 satisfying $F(z_0) \cap V_0 \neq \emptyset$, there exists a neighborhood G_0 of z_0 such that $F(z) \cap V_0 \neq \emptyset$ for all $z \in G_0$ (see [5, Definition 5.1.15, p. 173]). F is said to be *upper semicontinuous* at $z_0 \in E_1$ if for any open set V in E_2 satisfying $F(z_0) \subset V$, there exists a neighborhood G of z_0 such that $F(z) \subset V$ for all $z \in G$. If F is lower semicontinuous and upper semicontinuous at z_0, we say F is continuous at z_0.

Definition 2.1 *(a) The solution map $S : M \times \Lambda \rightrightarrows C([0,1], \mathbb{R}^n) \times L^p([0,1], \mathbb{R}^m)$ is said to be $(s,w)-$upper semicontinuous at $(\overline{\mu}, \overline{\lambda})$ if for any open set V_1 in $C([0,1], \mathbb{R}^n)$ and weakly open set V_2 in $L^p([0,1], \mathbb{R}^m)$ satisfying $S(\overline{\mu}, \overline{\lambda}) \subset V_1 \times V_2$, there exist a neighborhood U_1 of $\overline{\mu}$ and a neighborhood U_2 of $\overline{\lambda}$ such that*

$$S(\mu, \lambda) \subset V_1 \times V_2, \forall (\mu, \lambda) \in U_1 \times U_2.$$

(b) S is said to be $(s,w)-$lower semicontinuous at $(\overline{\mu}, \overline{\lambda})$ if for any open set V_1' in $C([0,1], \mathbb{R}^n)$ and weakly open set V_2' in $L^p([0,1], \mathbb{R}^m)$ satisfying $S(\overline{\mu}, \overline{\lambda}) \cap (V_1' \times V_2') \neq \emptyset$, there exist a neighborhood U_1' of $\overline{\mu}$ and a neighborhood U_2' of $\overline{\lambda}$ such that

$$S(\mu, \lambda) \cap (V_1' \times V_2') \neq \emptyset, \forall (\mu, \lambda) \in U_1' \times U_2'.$$

If S is both $(s,w)-$upper semicontinuous at $(\overline{\mu}, \overline{\lambda})$ and $(s,w)-$lower semicontinuous at $(\overline{\mu}, \overline{\lambda})$, then S is called $(s,w)-$continuous at $(\overline{\mu}, \overline{\lambda})$.

In Definition 2.1, if V_2 and V_2' are strongly open sets of $L^p([0,1], \mathbb{R}^m)$, we say S is $(s,s)-$upper semicontinuous and $(s,s)-$lower semicontinuous at $(\overline{\mu}, \overline{\lambda})$, respectively. It is clear that if S is $(s,s)-$ upper semicontinuous at $(\overline{\mu}, \overline{\lambda})$ then S is $(s,w)-$ upper semicontinuous at $(\overline{\mu}, \overline{\lambda})$. This implication is also true for lower semiconinuity of S.

In the sequel, we need the following assumptions on f, A, B and T.

$(H1)$ $f(\cdot, x, u, \mu)$ is a Carathéodory function, that is, for a.e. $t \in [0,1]$, $f(t, \cdot, \cdot, \cdot)$ is continuous in (x, u, μ) and for each fixed $(x, u, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r$, the function $f(\cdot, x, u, \mu)$ is measurable on $[0,1]$.

$(H2)$ Growth and dominated condition: there exist constants $\alpha_i \geq 0$ with $i = 1, 2, 3$ and a nonnegative function $\vartheta \in L^1([0,1], \mathbb{R})$ such that

$$|f(t, x, u, \mu)| \leq \vartheta(t) + \alpha_1 |x|^{\beta_1} + \alpha_2 |u|^{\beta_2} + \alpha_3 |\mu|^{\beta_3},$$

where $0 \leq \beta_1$, $1 \leq \beta_2 \leq p$, $1 \leq \beta_3 \leq r$ and $0 \leq \beta_3$ when $r = \infty$.

$(H3)$ Convexity: the function $u \mapsto f(t, x, u, \mu)$ is convex for all $(t, x, \mu) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^r$.

$(H4)$ The entries of $A(t,x)$ and $B(t,x)$ are continuous and continuously differentiable in x such that $A_x(\cdot, \cdot)$ and $B_x(\cdot, \cdot)$ are continuous. Also, the entries of $T(t, \lambda)$ are continuous.

Besides, there exist nonnegative functions $\phi \in L^1([0,1],\mathbb{R})$, $\psi \in L^q([0,1],\mathbb{R})$ and $\chi \in L^{s'}([0,1],\mathbb{R})$ such that

$$|A(t,x_1) - A(t,x_2)| \leq \phi(t)|x_1 - x_2|, \text{ a.e. } t \in [0,1], \forall x_1, x_2 \in \mathbb{R}^n, \tag{6}$$

$$|B(t,x_1) - B(t,x_2)| \leq \psi(t)|x_1 - x_2|, \text{ a.e. } t \in [0,1], \forall x_1, x_2 \in \mathbb{R}^n, \tag{7}$$

$$|T(t,\lambda_1) - T(t,\lambda_2)| \leq \chi(t)|\lambda_1 - \lambda_2|, \text{ a.e. } t \in [0,1], \forall \lambda_1, \lambda_2 \in \mathbb{R}^l. \tag{8}$$

Here q and s' are conjugate numbers of p and s, respectively. The norm of $n \times m$ matrix $B(t,x) = [b_{ij}(t,x)]$ is defined by $|B(t,x)|^2 = \sum_{i=1}^n \sum_{j=1}^m |b_{ij}(t,x)|^2$.

We are ready to state our main results.

Theorem 2.1 *Suppose that assumptions $(H1) - (H4)$ are fulfilled. Then the following assertions are valid:*
(i) $S(\mu,\lambda) \neq \emptyset$ for all $(\mu,\lambda) \in M \times \Lambda$;
(ii) $S(\cdot,\cdot)$ is $(s,w)-$upper semicontinuous at $(\overline{\mu},\overline{\lambda})$.

From Theorem 2.1 one may ask whether the solution map $S(\cdot,\cdot)$ is $(s,s)-$upper semicontinuous. The next theorem says that if the unperturbed problem is good enough and the space of parameters μ is good enough, then the solution map is $(s,s)-$upper semicontinuous and (s,s)-continuous at $(\overline{\mu},\overline{\lambda})$. For this we need the following strengthened assumption.

$(H5)$ Assume that $r = \infty$ and the function $(x,u) \mapsto L(t,x,u,\mu)$ is Fréchet continuously differentiable for a.e. $t \in [0,1]$ and $\mu \in \overline{\mu}(t) + \epsilon B_k(0,1)$ for some $\epsilon > 0$, where $B_k(0,1)$ is the unit ball in $\mathbb{R}^k$. Furthermore, the following conditions are fulfilled:
(i) There exist a continuous function $k_i : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$, positive numbers s_i with $i = 1,2$, $0 \leq \eta \leq p$ and $0 \leq \theta \leq p/q$ such that

$$|f_x(t,x,u,\mu) - f_x(t,x,u,\overline{\mu}(t))| \leq k_1(t,|x|,|\mu|,|\overline{\mu}(t)|)|u|^\eta |\mu - \overline{\mu}(t)|^{s_1} \tag{9}$$

and

$$|f_u(t,x,u,\mu) - f_u(t,x,u,\overline{\mu}(t))| \leq k_2(t,|x|,|\mu|,|\overline{\mu}(t)|)|u|^\theta |\mu - \overline{\mu}(t)|^{s_2} \tag{10}$$

for a.e. $t \in [0,1]$, $x \in \mathbb{R}^n$, $u \in [a(t),b(t)]$ and $\mu \in \overline{\mu}(t) + \epsilon B_k(0,1)$.
(ii) There exists a nonnegative function $k_3(\cdot) \in L^1([0,1],\mathbb{R})$ such that

$$|f_x(t,x_1,u_1,\overline{\mu}(t)) - f_x(t,x_2,u_2,\overline{\mu}(t))| \leq k_3(t)|x_1 - x_2| \tag{11}$$

for a.e. $t \in [0,1]$ and for all $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ with $i = 1,2$.
(iii) There exists a positive number α such that for any $(\hat{x},\hat{u}) \in S(\overline{\mu},\overline{\lambda})$ one has

$$\langle f_u(t,x,v,\overline{\mu}(t)) - f_u(t,\hat{x}(t),\hat{u}(t),\overline{\mu}(t)), v - \hat{u}(t)\rangle \geq \alpha |v - \hat{u}(t)|^p \tag{12}$$

for a.e. $t \in [0,1]$, for all $v \in [a(t),b(t)]$ and $x \in \mathbb{R}^n$.

Under this extra assumption, we have

Theorem 2.2 *Suppose that assumptions $(H1)-(H5)$ are fulfilled. Then the mapping $S(\cdot,\cdot)$ is $(s,s)-$upper semicontinuous at $(\overline{\mu},\overline{\lambda})$. Moreover if $S(\overline{\mu},\overline{\lambda})$ is a singleton, then $S(\cdot,\cdot)$ is $(s,s)-$continuous at $(\overline{\mu},\overline{\lambda})$.*

Notice that assumptions $(H1)-(H3)$ in Theorem 2.1 ensure that $J(\cdot,\cdot,\mu)$ is weakly lower semicontinuous for each $\mu\in M$. While assumption $(H4)$ guarantees that for each $\lambda\in\Lambda$ and $u\in U$, the state equation has a unique global solution $x\in W^{1,1}([0,1],\mathbb{R}^n)$. Condition (ii) in $(H5)$ says that f_x is a Lipschitz function which depends only on x. Condition (iii) in $(H5)$ requires that the function $f(t,x,\cdot,\overline{\mu}(t))$ is strongly convex in u. We can give several examples under which assumptions $(H1)-(H5)$ are fulfilled as follows.

Example 2.1 Let $n=m=k=l=1$ and $p=r=s=2$. Then problem $P(\mu,\lambda)$ with

$$\begin{aligned}&f(t,x,u,\mu)=x^3+u^2+\mu u,\\&A(t,x)=t+\sqrt{1+x^2},\ B(t,x)=tx, T(t,\lambda)=\lambda\end{aligned}$$

satisfies all assumptions $(H1)-(H4)$.

In order to verify $(H4)$ for $A(t,x)$ we use the Lagrange Theorem. Then for all $x,y\in\mathbb{R}$, we have

$$|A(t,x)-A(t,y)|=|\sqrt{1+x^2}-\sqrt{1+y^2}|\leq\frac{|\xi|}{\sqrt{1+\xi^2}}|x-y|\leq|x-y|,$$

where $\xi=\theta x+(1-\theta)y$ with $\theta\in[0,1]$.

Example 2.2 Let $m=n=k=l=2$ and $p,r,s\in(1,\infty)$. Assume that

$$\begin{aligned}&f(t,x,u,\mu)=x_1\mu_1+x_2\mu_2-x_1^2-x_2^2+|u|^p,\\&A(t,x)=\begin{pmatrix}x_1\\ \sin x_2\end{pmatrix}, B(t,x)=\begin{pmatrix}1 & x_2\\ \sin x_1 & 0\end{pmatrix}, T(t,\lambda)=\begin{pmatrix}t\lambda_1\\ t^2\lambda_2\end{pmatrix},\end{aligned}$$

where $x=(x_1,x_2),\lambda=(\lambda_1,\lambda_2)$ and $\mu=(\mu_1,\mu_2)$. Then $P(\mu,\lambda)$ satisfies assumptions $(H1)-(H4)$.

In order to verify $(H3)$ we use Young's Inequality:

$$|f(t,x,u,\mu)|\leq|x||\mu|+|u|^p\leq\frac{1}{r'}|x|^{r'}+\frac{1}{r}|\mu|^r+|u|^p, \tag{13}$$

where $\frac{1}{r'}+\frac{1}{r}=1$ and $x=(x_1,x_2)$.

Example 2.3 Let $m=n=k=l=1$ and $p=2,s=1,r=\infty$. We consider the problem

$$\begin{cases}J(x,u,\mu)=\int_0^1\big((u(t)-\mu(t))^2-\frac{1}{2}x^2(t)-\mu(t)u(t)x(t)\big)dt\to\inf,\\ \dot{x}(t)=u(t)+\lambda(t),\\ x(0)=0,\\ -1\leq u(t)\leq 1.\end{cases} \tag{14}$$

Here we assume that $\overline{\mu}(t) = 0, \overline{\lambda}(t) = 0$ for all $t \in [0,1]$. From the above, we can verify that assumptions $(H1)-(H5)$ of the Theorem 2.1 are fulfilled. In order to check $(H5)$ we notice that $f(t,x,u,\mu) = (u-\mu)^2 - \frac{1}{2}x^2 - \mu x u$, $f_x = x, f_u = 2(u-\mu) - \mu x$. Hence conditions (i) and (ii) in $(H5)$ are valid. For condition (iii), we have

$$\langle f_u(t,x,u,\overline{\mu}) - f_u(t,\overline{x},\overline{u},\overline{\mu}), u - \overline{u}\rangle = 2|u-\overline{u}|^2. \tag{15}$$

Note that J is convex in u and concave in x. We now assume that $(\overline{x}(\mu,\mu), \overline{u}(\mu,\mu)) \in S(\overline{\mu}, \overline{\lambda})$. Then it must satisfy the Pontryagin Maximum Principle. According to the Pontryagin Maximum Principle (see [10, Theorem 1, p. 134 and p. 139]), there exists an absolute continuous function $\phi(t)$ such that the following conditions are valid:
(i) the adjoint equation:

$$\begin{cases} \dot{\phi} = -\overline{x}, \\ \phi(1) = 0. \end{cases} \tag{16}$$

(ii) the maximum principle:

$$\phi(t)\overline{u}(t) - \overline{u}^2(t) + \frac{1}{2}\overline{x}^2(t) = \max_{-1\le u\le 1}\big(\phi(t)u - u^2 + \frac{1}{2}\overline{x}^2(t)\big),$$

from which we see that

$$\overline{u}(t) = \begin{cases} \frac{\phi(t)}{2} & \text{if } -1 \le \frac{\phi(t)}{2} \le 1, \\ -1 & \text{if } \frac{\phi(t)}{2} < -1, \\ 1 & \text{if } \frac{\phi(t)}{2} > 1. \end{cases}$$

From the state equation, we have $\overline{x}(t) = \int_0^t \overline{u}(s)dt$. This implies that

$$|\overline{x}(t)| \le \int_0^1 |\overline{u}(s)|ds \le 1.$$

On the other hand, from the adjoint equation, we have $\phi(t) = -\int_1^t \overline{x}(s)ds$. It follows that

$$|\phi(t)| \le \int_0^1 |\overline{x}(s)|ds \le 1, \ \forall t \in [0,1].$$

Therefore we have $\overline{u}(t) = \frac{\phi(t)}{2}$. Combining this with the adjoint equation, yields

$$\dot{\phi}(t) = -\int_0^t \overline{u}(s)ds = -\frac{1}{2}\int_0^t \phi(s)ds.$$

It follows that

$$\begin{cases} \ddot{\phi}(t) = -\frac{1}{2}\phi(t), \\ \dot{\phi}(0) = 0, \ \phi(1) = 0. \end{cases}$$

Hence $\phi(t) = a\cos\frac{t}{\sqrt{2}} + b\sin\frac{t}{\sqrt{2}}$ and so $\phi(t) = 0$ for all $t \in [0,1]$. Consequently, $\overline{u}(t) = 0$, $\overline{x}(t) = 0$ and $S(\overline{\mu}, \overline{\lambda}) = \{(0,0)\}$. By Theorem 2.2, $S(\mu,\lambda)$ is continuous at $(0,0)$.

To end this section, we give an example showing that although the unperturbed problem has a unique solution, the perturbed problems may have several solutions and the solution map is continuous at a reference point.

Example 2.4 Suppose that $m = n = l = k = 1$, $p = 4$, $r = s = \infty$ and $(\overline{\mu}, \overline{\lambda}) = (0, 0)$. We consider problem $P(\mu, \lambda)$ of finding $u \in L^4([0, 1], \mathbb{R})$ and $y \in W^{1,1}([0, 1], \mathbb{R})$ such that

$$J(x, u, \mu) = \int_0^1 f(t, x(t), u(t), \mu(t))dt \to \inf \tag{17}$$

with the state equation

$$\dot{x} = x + xu + \lambda, \ x(0) = 1 \tag{18}$$

and pointwise constraints

$$0 \le u(t) \le 1, \tag{19}$$

where f is given by

$$f(t, x, u, \mu) = \frac{1}{2}\big[1 - \mathrm{sign}(u + \mu^2)\big](u + \mu^2)^4 + \frac{1}{2}\big[1 + \mathrm{sign}(u - \mu^2)\big](u - \mu^2)^4.$$

Here $\mathrm{sign}(u)$ is defined by

$$\mathrm{sign}(u) = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -1 & \text{if } u < 0. \end{cases}$$

Then we have the following assertions:
(i) $P(\overline{\mu}, \overline{\lambda})$ has unique solution $(\overline{x}, \overline{u}) = (e^t, 0)$.
(ii) If $0 < |\mu(t)| \le 1$, then we have

$$S(\mu, \lambda) \supset \Big\{(x(\mu, \lambda), s\mu^2(t)), 0 < s < 1\Big\},$$

where $x(\mu, \lambda)$ is solutions of the equation:

$$\dot{x}(t) = x(t) + sx(t)\mu^2(t) + \lambda(t), \ x(0) = 1. \tag{20}$$

In fact, when $\overline{\mu} = \overline{\lambda} = 0$, problem $P(0, 0)$ becomes

$$J(u, \overline{\mu}) = \int_0^1 u^4(t)dt \to \inf$$

with constraints

$$\begin{cases} \dot{x} = x + u \\ x(0) = 1, \\ 0 \le u(t) \le 1. \end{cases}$$

Obviously, $S(0,0) = \{(e^t, 0)\}$. We now show that $P(\mu, \lambda)$ satisfies $(H1) - (H5)$. It is easy to see that

$$f(t,x,u,\mu) = \begin{cases} (u+\mu^2)^4 & \text{if } u < -\mu^2 \\ 0 & \text{if } -\mu^2 \leq u \leq \mu^2 \\ (u-\mu^2)^4 & \text{if } u > \mu^2. \end{cases}$$

Hence

$$f_u(t,x,u,\mu) = \begin{cases} 4(u+\mu^2)^3 & \text{if } u < -\mu^2 \\ 0 & \text{if } -\mu^2 \leq u \leq \mu^2 \\ 4(u-\mu^2)^3 & \text{if } u > \mu^2. \end{cases}$$

and $f_x(t,x,u,\mu) = 0$. Hence assumptions $(H1) - (H4)$ are satisfied. In order to verify $(H5)$ we need to check conditions (10) and (12). For all $x \in \mathbb{R}$ and $u \in [0,1]$, we have

$$f_u(t,x,u,\mu) - f_u(t,x,u,0) = \begin{cases} 4(u+\mu^2)^3 - 4u^3 & \text{if } u \leq -\mu^2 \\ -4u^3 & \text{if } -\mu^2 \leq u \leq \mu^2 \\ 4(u-\mu^2)^3 - 4u^3 & \text{if } u \geq \mu^2. \end{cases}$$

Hence for all $x \in \mathbb{R}$, $0 \leq u \leq 1$ and $|\mu| \leq 1$, we have

$$\begin{aligned} &\left|f_u(t,x,u,\mu) - f_u(t,x,u,0)\right| \leq 4(\mu^2)^3 + |4(u-\mu^2)^3 - 4u^3| \\ &\leq 4\mu^6 + 12u^2\mu + 12|u|\mu^2 + 4\mu^6 \leq 32|\mu|. \end{aligned}$$

Consequently, (10) is valid. Also, for all $u_1, u_2 \in [0,1]$ and $x_1, x_2 \in \mathbb{R}$, we have

$$\begin{aligned} &(f_u(t,x_1,u_1,0) - f_u(t,x_2,u_2,0))(u_1-u_2) \\ &= 4(u_1^3 - u_2^3)(u_1-u_2) = 4(u_1-u_2)^2(u_1^2+u_2^2+u_1u_2) \\ &= 4(u_1-u_2)^2((u_1-u_2)^2 + 3u_1u_2) \geq 4(u_1-u_2)^4. \end{aligned}$$

Thus condition (12) is fulfilled. Finally, if $u(\mu,\lambda) = s\mu^2$ with $0 < s < 1$, then $J(u,\mu) = 0$. Let $x(\mu,\lambda)$ be solutions of (20). Then $(x(\mu,\lambda), u(\mu,\lambda)) \in S(\mu,\lambda)$.

3 Auxiliary results

The following lemma gives existence of global solution of (2).

Lemma 3.1 *Suppose that assumption $(H4)$ is fulfilled. Then for each $u \in L^p([0,1], \mathbb{R}^m)$ and $\lambda \in L^s([0,1], \mathbb{R}^l)$, equation (2) has a unique solution $x \in W^{1,1}([0,1], \mathbb{R}^n)$.*

Proof. Consider the mapping

$$F(x)(t) = x_0 + \int_0^t \big(A(s,x(s)) + B(s,x(s))u(s) + T(s,\lambda(s))\big)ds.$$

We show that F^j is a contraction mapping from $C([0,1],\mathbb{R}^n)$ into itself for j big enough. We put $\omega(t)=\phi(t)+\psi(t)|u(t)|$. Then $\omega\in L^1([0,1],\mathbb{R})$ and for all $x_1,x_2\in C([0,1],\mathbb{R}^n)$, we have

$$
\begin{aligned}
&|(F(x_1)-F(x_2))(t)|\\
&=\Big|\int_0^t \big(A(s,x_1(s))-A(s,x_2(s))+[B(s,x_1(s))-B(s,x_2(s))]u(s)\big)ds\Big|\\
&\le \int_0^t \big(\big|(A(s,x_1(s))-A(s,x_2(s))\big|+\big|[B(s,x_1(s))-B(s,x_2(s))]u(s)\big|\big)ds\\
&\le \int_0^t \big(\phi(s)|x_1(s)-x_2(s)|+\psi(s)|x_1(s)-x_2(s)||u(s)|\big)ds\\
&=\int_0^t \omega(s_1)|x_1(s_1)-x_2(s_1)|ds_1.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
|(F^2(x_1)-F^2(x_2))(t)| &\le \int_0^t \omega(s_1)|F(x_1)(s_1)-F(x_2)(s_1)|ds_1\\
&\le \int_0^t \omega(s_1)ds_1\int_0^{s_1}\omega(s_2)|x_1(s_2)-x_2(s_2)|ds_2.
\end{aligned}
$$

Continuing the process, we get

$$
\begin{aligned}
&|(F^j(x_1)-F^j(x_2))(t)| \le \int_0^t \omega(s_1)|F^{j-1}x_1(s_1)-F^{j-1}x_2(s_1)|ds_1\\
&\le \int_0^t ds_1\omega(s_1)\int_0^{s_1}ds_2\omega(s_2)\cdots\int_0^{s_{j-1}}ds_j\omega(s_j)|x_1(s_j)-x_2(s_j)|\\
&\le \|x_1-x_2\|_0\int_0^t ds_1\omega(s_1)\int_0^{s_1}ds_2\omega(s_2)\cdots\int_0^{s_{j-1}}ds_j\omega(s_j).
\end{aligned}
$$

By induction, we can show that

$$
\int_0^t ds_1\omega(s_1)\int_0^{s_1}ds_2\omega(s_2)\cdots\int_0^{s_{j-1}}ds_j\omega(s_j)=\frac{1}{j!}\Big(\int_0^t\omega(s)ds\Big)^j.
$$

Consequently, we have

$$
|(F^j(x_1)-F^j(x_2))(t)|\le \frac{1}{j!}\Big(\int_0^t\omega(s)ds\Big)^j\|x_1-x_2\|_0\le \frac{1}{j!}\Big(\int_0^1\omega(s)ds\Big)^j\|x_1-x_2\|_0.
$$

Hence

$$
|F^j(x_1)-F^j(x_2)|_0\le \frac{1}{j!}\Big(\int_0^1\omega(s)ds\Big)^j\|x_1-x_2\|_0.
$$

Since $\frac{1}{j!}\big(\int_0^1\omega(s)ds\big)^j<1$ when j is sufficiently large, we see that F^j is a contraction mapping. By the Contraction Mapping Theorem, there exists a unique $x\in C([0,1],\mathbb{R}^n)$

such that $F^j(x) = x$. By the Contraction Mapping Principle in [10, Chapter 0, p.13] (see also [14, Lemma 5.4.3, p. 323]), x is also a fixed point of F, that is,

$$x(t) = x_0 + \int_0^t \big(A(s, x(s)) + B(s, x(s))u(s) + T(s, \lambda(s))\big)ds.$$

By $(H4)$, we have

$$\begin{aligned}&|A(s, x(s)) + B(s, x(s))u(s) + T(s, \lambda(s))|\\ &\le \phi(s)|x(s)| + |A(s, 0)| + (\psi(s)|x(s)| + |B(s, 0)|)|u(s)| + \chi(s)|\lambda(s)| + |T(s, 0)|.\end{aligned}$$

It is easy to see that the function in right hand side belongs to $L^1([0, 1], \mathbb{R})$. Hence

$$|A(\cdot, x(\cdot)) + B(\cdot, x(\cdot))u(\cdot) + T(\cdot, \lambda(\cdot))| \in L^1([0, 1], \mathbb{R}).$$

It follows that $x \in W^{1,1}([0, 1], \mathbb{R}^n)$ and

$$\begin{cases}\dot{x}(t) = A(t, x(t)) + B(t, x(t))u(t) + T(t, \lambda(t)), \text{a.e. } t \in [0, 1],\\ x(0) = x_0.\end{cases}$$

The proof of the lemma is complete. □

The following lemma shows that $K(\cdot)$ has Lipschitz property.

Lemma 3.2 *Suppose that assumption* $(H4)$ *is fulfilled. Then the set-valued map* $K(\cdot)$ *which is defined by* (4), *has closed values and there exists a constant* $k_0 > 0$ *such that*

$$K(\lambda_1) \subset K(\lambda_2) + k_0\|\lambda_1 - \lambda_2\|_s \overline{B}_Z, \quad \forall \lambda_1, \lambda_2 \in \Lambda. \tag{21}$$

Proof. Let $z_i = (x_i, u_i) \in K(\lambda)$ such that $z_i \to z = (x, u)$ as $i \to \infty$. Then $x_i \to x$ uniformly, $\dot{x}_i \to \dot{x}$ and $u_i \to u$ strongly in L^1. By passing to subsequence if necessary, we may assume that $\dot{x}_i \to \dot{x}$ and $u_i \to u$ almost everywhere in $t \in [0, 1]$. Note that

$$\begin{cases}\dot{x}_i(t) = A(t, x_i(t)) + B(t, x_i(t))u_i(t) + T(t, \lambda(t))\\ x_i(0) = x_0.\end{cases}$$

By letting $i \to \infty$ and using the fact that the entries of $A(t, x)$ and $B(t, x)$ are continuous, we obtain

$$\begin{cases}\dot{x}(t) = A(t, x(t)) + B(t, x(t))u(t) + T(t, \lambda(t))\\ x(0) = x_0.\end{cases}$$

Besides, we have $a(t) \le u(t) \le b(t)$ for a.e. $t \in [0, 1]$. Hence $z \in K(\lambda)$ and so $K(\lambda)$ is a closed set. It remains to prove that $K(\cdot)$ has Lipschitz property.

Fixing any $\lambda_1, \lambda_2 \in \Lambda$, we show that there exists a constant $k_0 > 0$ such that (21) is satisfied. By Lemma 3.1, $K(\lambda) \neq \emptyset$ for all $\lambda \in \Lambda$. Let $(x, u) \in K(\lambda_1)$. Then one has

$$\dot{x}(t) = A(t, x(t)) + B(t, x(t))u(t) + T(t, \lambda_1(t)), \text{ a.e. } t \in [0, 1]. \tag{22}$$

We have to prove that there exists $(y, v) \in K(\lambda_2)$ such that

$$\|(x,u)-(y,v)\| \le k_0\|\lambda_1-\lambda_2\|_s$$

for some absolute constant $k_0 > 0$. Taking $v = u$ and using Lemma 3.1, we see that there exists $y \in X$ such that

$$\begin{cases} \dot y(t) = A(t,y(t)) + B(t,y(t))u(t) + T(t,\lambda_2(t)),\ \forall t \in [0,1], \\ y(0) = x_0. \end{cases} \tag{23}$$

By subtracting (22) and (23) and putting $w = x - y$, we get $w(0) = 0$ and

$$\dot w = A(t,x(t)) - A(t,y(t)) + [B(t,x(t)) - B(t,y(t))]u(t) + T(t,\lambda_1) - T(t,\lambda_2). \tag{24}$$

From this and $(H4)$, we have

$$\begin{aligned} |\dot w| &\le \phi(t)|w(t)| + \psi(t)|w(t)||u(t)| + \chi(t)|\lambda_1(t)-\lambda_2(t)| \\ &\le |w(t)|\big[\phi(t) + \psi(t)(|a(t)|+|b(t)|)\big] + \chi(t)|\lambda_1(t)-\lambda_2(t)| \\ &\le |w(t)|\zeta(t) + \chi(t)|\lambda_1(t)-\lambda_2(t)|, \end{aligned} \tag{25}$$

where $\zeta(t) := \big[\phi(t) + \psi(t)(|a(t)|+|b(t)|)\big]$ which belongs to $L^1([0,1],\mathbb{R})$.

Since $w(t) = \int_0^t \dot w(s)ds$, we obtain

$$\begin{aligned} |w(t)| &\le \int_0^t \big(|w(s)|\zeta(s) + \chi(s)|\lambda_1(s)-\lambda_2(s)|\big)ds \\ &\le \int_0^t |w(s)|\zeta(s)ds + \int_0^1 \chi(s)|\lambda_1(s)-\lambda_2(s)|ds \\ &\le \int_0^t |w(s)|\zeta(s)ds + \|\chi(\cdot)\|_{s'}\|\lambda_1-\lambda_2\|_s. \end{aligned}$$

By Gronwall's Inequality (see [6, Lemma 18.1.i]), we obtain

$$|w(t)| \le \|\chi(\cdot)\|_{s'}\|\lambda_1-\lambda_2\|_s \exp\Big(\int_0^1 \zeta(s)ds\Big).$$

Combining this with (25), we have

$$|\dot w(t)| \le \|\chi(\cdot)\|_{s'} \exp\Big(\int_0^1 \zeta(s)ds\Big)\|\lambda_1-\lambda_2\|_s\zeta(t) + |\chi(t)||\lambda_1(t)-\lambda_2(t)|.$$

From this and Hölder's Inequality, we have

$$\|\dot w\|_1 \le \|\chi(\cdot)\|_{s'} \exp\Big(\int_0^1 \zeta(s)ds\Big)\|\zeta\|_1\|\lambda_1-\lambda_2\|_s + \|\chi(\cdot)\|_{s'}\|\lambda_1-\lambda_2\|_s.$$

Define

$$k_0 = \|\chi(\cdot)\|_{s'} \exp\Big(\int_0^1 \zeta(s)ds\Big)\|\zeta\|_1 + \|\chi(\cdot)\|_{s'}. \tag{26}$$

Then we have

$$\|(x,u)-(y,v)\| = \|x-y\|_{1,1} = \|w\|_{1,1} = |w(0)| + \|\dot w\|_1 \le k_0\|\lambda_1-\lambda_2\|_s.$$

The proof of the lemma is complete. □

Lemma 3.3 *Suppose that assumptions* $(H4)$ *is valid,* $\{\lambda_j\}$ *and* $\{(x_j, u_j)\}$ *are sequences in* Λ *and* Z*, respectively. Suppose that* $(x_j, u_j) \in K(\lambda_j)$*,* $\lambda_j \to \overline{\lambda}$ *strongly in* $L^s([0,1], \mathbb{R}^l)$*,* $x_j \to x$ *uniformly on* $[0,1]$*,* $\dot{x}_j \rightharpoonup \dot{x}$ *weakly in* $L^1([0,1], \mathbb{R}^n)$ *and* $u_j \rightharpoonup u$ *weakly in* $L^p([0,1], \mathbb{R}^m)$*. Then one has* $(x, u) \in K(\overline{\lambda})$*.*

Proof. By assumption, we have

$$\dot{x}_j(t) = A(t, x_j(t)) + B(t, x_j(t))u_j(t) + T(t, \lambda_j(t)). \tag{27}$$

In order to complete the proof, we need to show that

$$A(\cdot, x_j(\cdot)) + B(\cdot, x_j(\cdot))u_j(\cdot) + T(\cdot, \lambda_j(\cdot)) \rightharpoonup A(\cdot, x) + B(\cdot, x)u + T(\cdot, \overline{\lambda}) \tag{28}$$

weakly in $L^1([0,1], \mathbb{R}^n)$ when $j \to \infty$. In fact, by $(H4)$, we have

$$|A(t, x_j(t)) - A(t, x(t))| \le \phi(t)|x_j(t) - x(t)|. \tag{29}$$

It follows that

$$\|A(\cdot, x_j) - A(\cdot, x)\|_1 \le \|\phi\|_1 \|x_j - x\|_0 \to 0 \text{ as } j \to \infty.$$

Hence $A(\cdot, x_j) \to A(\cdot, x)$ strongly in $L^1([0,1], \mathbb{R}^n)$. Similarly, we have $T(\cdot, \lambda_j) \to T(\cdot, \overline{\lambda})$ strongly in $L^1([0,1], \mathbb{R}^n)$. It remains to prove that $B(\cdot, x_j(\cdot))u_j(\cdot) \rightharpoonup B(\cdot, x(\cdot))u$ weakly in $L^1([0,1], \mathbb{R}^n)$. For this we write

$$B(t, x_j(t))u_j(t) - B(t, x(t))u(t) = [B(t, x_j(t)) - B(t, x(t))]u_j(t) + B(t, x(t))(u_j(t) - u(t)). \tag{30}$$

By $(H4)$, we have

$$|[B(t, x_j(t)) - B(t, x(t))]u_j(t)| \le \psi(t)|x_j(t) - x(t)||u_j(t)|.$$

This implies that

$$\|(B(\cdot, x_j) - B(\cdot, x))u_j\|_1 \le \|\psi\|_q \|u_j\|_p \|x_j - x\|_0 \to 0 \text{ as } j \to \infty$$

because $\|u_j\|_p$ is bounded and $\|x_j - x\|_0 \to 0$. Hence $(B(\cdot, x_j) - B(\cdot, x))u_j \to 0$ strongly in $L^1([0,1], \mathbb{R}^n)$ and so $(B(\cdot, x_j) - B(\cdot, x))u_j \rightharpoonup 0$ weakly in $L^1([0,1], \mathbb{R}^n)$. For second term, we take the scalar product with any $\vartheta \in L^\infty([0,1], \mathbb{R}^n)$ and get

$$\int_0^1 (B(t, x(t))(u_j(t) - u(t)), \vartheta(t))dt = \int_0^1 (u_j(t) - u(t), B(t, x(t))^T \vartheta(t))dt,$$

where $B(t, x(t))^T$ is the transporse matrix of $B(t, x(t))$. By $(H4)$ we have

$$\begin{aligned} |B(t, x(t))^T \vartheta(t))| &\le |B(t, x(t))^T||\vartheta(t)| = |B(t, x(t))||\vartheta(t)| \\ &\le (\psi(t)|x(t)| + |B(t, 0)|)|\vartheta(t)|. \end{aligned}$$

This implies that $B(t,x(t))^T\vartheta(t) \in L^q([0,1],\mathbb{R}^m)$. Hence

$$\int_0^1 (B(t,x(t))(u_j(t)-u(t)),\vartheta(t))dt = \int_0^1 (u_j(t)-u(t), B(t,x(t))^T\vartheta(t))dt \to 0$$

as $j\to\infty$ because $u_j \rightharpoonup u$ in $L^p([0,1],\mathbb{R}^m)$. From (30), we get

$$B(\cdot,x_j(\cdot))u_j(\cdot) - B(\cdot,x(\cdot))u \rightharpoonup 0$$

weakly in $L^p([0,1],\mathbb{R}^n)$. In summary, assertion (28) is justified. By taking the limit on two sides of (27), we get

$$\dot{x}(t) = A(t,x(t)) + B(t,x(t))u(t) + T(t,\overline{\lambda}(t)).$$

Since $x_j \to x$ uniformly, we get $x(0)=x_0$. Since the set

$$\{v \in L^p([0,1],\mathbb{R}^m) : a(t) \le v(t) \le b(t)\}$$

is weakly closed, we get $a(t)\le u(t)\le b(t)$. Hence $(x,u)\in K(\overline{\lambda})$. The proof of the lemma is complete. □

4 Proof of the main result

• Proof of Theorem 2.1

(i) Existence. For each $(\mu,\lambda)\in M\times\Lambda$ we define

$$V(\mu,\lambda) = \inf_{(x,u)\in K(\lambda)} J(x,u,\mu). \tag{31}$$

By Lemma 3.1, $K(\lambda)\neq\emptyset$. Taking any $(x,u)\in K(\lambda)$, we have from $(H2)$ that

$$|f(t,x(t),u(t),\mu(t))| \le \vartheta(t) + \alpha_1|x(t)|^{\beta_1} + \alpha_2|u(t)|^{\beta_2} + \alpha_3|\mu(t)|^{\beta_3} \tag{32}$$

with $1\le\beta_2\le p$ and $1\le\beta_3\le r$. This implies that

$$V(\mu,\lambda) \le J(x,u,\mu) \le \|\vartheta\|_1 + C_1\|x\|_0^\alpha + C_2\|u\|_p^p + C_3\|\mu\|_r^r < +\infty$$

for some constants $C_i>0$, $i=1,2,3$. By definition, there exists a sequence $(x_j,u_j)\in K(\lambda)$ such that

$$V(\mu,\lambda) = \lim_{j\to\infty} J(x_j,u_j,\mu). \tag{33}$$

Since $(x_j,u_j)\in K(\lambda)$, we have

$$\begin{cases} \dot{x}_j(t) = A(t,x_j(t)) + B(t,x_j(t))u_j(t) + T(t,\lambda(t)) \\ x_j(0) = x_0 \end{cases} \tag{34}$$

and $|u_j(t)| \le |a(t)| + |b(t)|$. By $(H4)$ we have

$$\begin{aligned}|\dot{x}_j(t)| &\le \phi(t)|x_j(t)| + |A(t,0)| + (\psi(t)|x_j(t)| + |B(t,0)|)|u_j(t)| + |\chi(t)||\lambda(t)| + |T(t,0)| \\ &= |x_j(t)|(\phi(t) + \psi(t)|u_j(t)|) + |A(t,0)| + |B(t,0)||u_j(t)| + \chi(t)|\lambda(t)| + |T(t,0)|.\end{aligned} \tag{35}$$

Since $x_j(t) = x_0 + \int_0^t \dot{x}_j(s)ds$, we get

$$\begin{aligned}|x_j(t)| \le & |x_0| + \int_0^t (\phi(s) + \psi(s)(|a(s)| + |b(s)|))|x_j(s)|ds \\ & + \int_0^t \big(|A(s,0)| + |B(s,0)|(|a(s)| + |b(s)|) + \chi(s)|\lambda(s)| + |T(s,0)|\big)ds \\ & \le \int_0^t (\phi(s) + \psi(s)(|a(s)| + |b(s)|))|x_j(s)|ds \\ & + |x_0| + \int_0^1 \big(|A(s,0)| + |B(s,0)|(|a(s)| + |b(s)|) + \chi(s)|\lambda(s)| + |T(s,0)|\big)ds.\end{aligned}$$

Define

$$\begin{aligned}&\gamma_1(t) = \phi(t) + \psi(t)(|a(t)| + |b(t)|), \\ &\gamma_2(t) = |A(t,0)| + |B(t,0)|(|a(t)| + |b(t)|) + \chi(t)|\lambda(t)| + |T(t,0)|, \\ &M_1 = |x_0| + \int_0^1 \big(|A(s,0)| + |B(s,0)|(|a(s)| + |b(s)|) + \chi(s)|\lambda(s)| + |T(s,0)|\big)ds.\end{aligned}$$

Then $\gamma_1, \gamma_2 \in L^1([0,1], \mathbb{R})$ and we have

$$|x_j(t)| \le \int_0^t \gamma_1(s)|x_j(s)|ds + M_1.$$

By Grownwall's Inequality (see [6, Lemma 18.1.i]) we get

$$|x_j(t)| \le M_1 \exp\Big(\int_0^1 \gamma_1(s)ds\Big) := M_2. \tag{36}$$

Hence $\|x_j\|_0$ is bounded. From this and (35), we obtain

$$|\dot{x}_j| \le M_2\gamma_1(t) + \gamma_2(t). \tag{37}$$

Hence

$$\|\dot{x}_j\|_1 \le M_2\|\gamma_1\|_1 + \|\gamma_2\|_1. \tag{38}$$

Besides, if E is a measurable set of $[0,1]$, then form (37), we have

$$\int_E |\dot{x}_j(t)|dt \le M_2 \int_E \gamma_1(t)dt + \int_E \gamma_2(t)dt. \tag{39}$$

It is clear that the right hand side of (39) approaches to 0 as $|E| \to 0$. Hence $\{\dot{x}_j\}$ is equiabsolutely integrable. From this and [6, Theorem 10.2.i, p. 317], $\{x_j\}$ is equiabsolutely continous. By Ascoli's Theorem, $\{x_j\}$ is a relatively compact set in $C([0,1],\mathbb{R}^n)$. Hence, by passing to a subsequence if necessary, we can assume that $x_j \to \hat{x}$ uniform in $[0,1]$. On the other hand $\{\dot{x}_j\}$ is bounded and equiabsolutely integrable. The Dunford-Pettis Theorem (see [6, Theorem 10.3.i]) implies that there exists a function $\xi \in L^1([0,1],\mathbb{R}^n)$ such that $\dot{x}_i \rightharpoonup \xi$ weakly in L^1. Since $x_j(t) = x_0 + \int_0^t \dot{x}_j(s)ds$, we obtain $\hat{x} = x_0 + \int_0^t \xi(s)ds$ and so $\dot{\hat{x}}(t) = \xi(t)$ a.e. We now notice that $|u_j(t)| \le |a(t)| + |b(t)|$. Hence $\{u_j\}$ is bounded in $L^p([0,1],\mathbb{R}^m)$. Without loss of generality, we may assume that $u_j \rightharpoonup \hat{u}$ for some $\hat{u} \in L^p([0,1],\mathbb{R}^m)$. By Lemma 3.3, we obtain $(\hat{x},\hat{u}) \in K(\overline{\lambda})$.

By $(H1),(H2)$ and $(H3)$, J is weakly lower semicontinuous (see [6, Theorem 2.18.i, Theorem 10.8.i] and [7, Theorem 3.3, p. 84]). Hence from (33), we have

$$V(\mu,\lambda) = \lim_{j\to\infty} J(x_j,u_j,\mu) \ge J(\hat{x},\hat{u},\mu).$$

This implies that $(\hat{x},\hat{u}) \in S(\mu,\lambda)$.

(ii) Upper semicontinuity of $S(\cdot,\cdot)$.

Assume that V_1 is an open set in $C([0,1],\mathbb{R}^n)$ and V_2 is a weakly open set in $L^p([0,1],\mathbb{R}^m)$ such that

$$S(\overline{\mu},\overline{\lambda}) \subset V_1 \times V_2 := V. \tag{40}$$

We want to show that there exists a neighborhood $M_0 \times \Lambda_0$ of $(\overline{\mu},\overline{\lambda})$ such that

$$S(\mu,\lambda) \subset V, \forall (\mu,\lambda) \in M_0 \times \Lambda_0. \tag{41}$$

By contradiction, we find out a sequence $(\mu_i,\lambda_i) \to (\overline{\mu},\overline{\lambda})$ strongly in $L^r([0,1],\mathbb{R}^k) \times L^s([0,1],\mathbb{R}^l)$ and a sequence $(x_i,u_i) \in S(\mu_i,\lambda_i)$ such that $(x_i,u_i) \notin V$. If we can show that there exists a subsequence $\{(x_{i_j},u_{i_j})\}$ of $\{(x_i,u_i)\}$ such that $x_{i_j} \to \overline{x}$ uniformly on $[0,1]$ and $u_{i_j} \rightharpoonup \overline{u}$ weakly in $L^p([0,1],\mathbb{R}^m)$ for some $(\overline{x},\overline{u}) \in S(\overline{\mu},\overline{\lambda})$, then $(x_{i_j},u_{i_j}) \in V$ for j large enough. This leads to a contradiction and the proof is completed. Therefore, it remains to prove the following lemma.

Lemma 4.1 *There exists $(\overline{x},\overline{u}) \in S(\overline{\mu},\overline{\lambda})$ and a subsequence $\{(x_{i_j},u_{i_j})\}$ of $\{(x_i,u_i)\}$ such that $x_{i_j} \to \overline{x}$ uniformly on $[0,1]$ and $u_{i_j} \rightharpoonup \overline{u}$ weakly in $L^p([0,1],\mathbb{R}^m)$ as $j \to \infty$.*

Proof. Since $(x_i,u_i) \in S(\mu_i,\lambda_i)$, $(x_i,u_i) \in K(\lambda_i)$. Hence

$$\begin{cases} \dot{x}_i(t) = A(t,x_i(t)) + B(t,x_i(t))u_i(t) + T(t,\lambda_i(t)) \\ x_i(0) = x_0 \end{cases} \tag{42}$$

and $|u_i(t)| \leq |a(t)| + |b(t)|$. By $(H4)$ we have

$$\begin{aligned}|\dot{x}_i(t)| &\leq \phi(t)|x_i(t)| + |A(t,0)| + (\psi(t)|x_i(t)| + |B(t,0)|)|u_i(t)| + \chi(t)|\lambda_i(t)| + |T(t,0)| \\ &= |x_i(t)|(\phi(t) + \psi(t)|u_i(t)|) + |A(t,0)| + |B(t,0)||u_i(t)| + \chi(t)|\lambda_i(t)| + |T(t,0)|.\end{aligned}$$

Since $\lambda_i \to \overline{\lambda}$ strongly in $L^s([0,1],\mathbb{R}^l)$, by passing to a subsequence if necessary, there exists a function $\gamma \in L^s([0,1],\mathbb{R})$ such that $|\lambda_i(t)| \leq \gamma(t)$ for a.e. $t \in [0,1]$ (see [7, Theorem 1.20]). It follows that

$$\begin{aligned}|\dot{x}_i(t)| \leq\, &|x_i(t)|(\phi(t) + \psi(t)(|a(t)| + |b(t)|)) \\ &+ |A(t,0)| + |B(t,0)|(|a(t)| + |b(t)|) + \chi(t)|\gamma(t)|.\end{aligned} \tag{43}$$

Since $x_i(t) = x_0 + \int_0^t \dot{x}_i(s)ds$, we get

$$\begin{aligned}|x_i(t)| \leq &|x_0| + \int_0^t (\phi(s) + \psi(s)(|a(s)| + |b(s)|))|x_j(s)|ds \\ &+ \int_0^t \big(|A(s,0)| + |B(s,0)|(|a(s)| + |b(s)|) + |\chi(s)|\gamma(s)\big)ds \\ &\leq \int_0^t (\phi(s) + \psi(s)(|a(s)| + |b(s)|))|x_i(s)|ds \\ &+ |x_0| + \int_0^1 \big(|A(s,0)| + |B(s,0)|(|a(s)| + |b(s)|) + \chi(s)|\gamma(s)| + |T(s,0)|\big)ds.\end{aligned}$$

Define

$$\begin{aligned}\hat{\gamma}_1(t) &= \phi(t) + \psi(t)(|a(t)| + |b(t)|), \\ \hat{\gamma}_2(t) &= |A(t,0)| + |B(t,0)|(|a(t)| + |b(t)|) + \chi(t)\gamma(t) + |T(t,0)|, \\ \widehat{M}_1 &= |x_0| + \int_0^1 \big(|A(s,0)| + |B(s,0)|(|a(s)| + |b(s)|) + \chi(s)\gamma(s) + |T(s,0)|\big)ds.\end{aligned}$$

Then $\hat{\gamma}_1, \hat{\gamma}_2 \in L^1([0,1],\mathbb{R})$ and we have

$$|x_i(t)| \leq \int_0^t \hat{\gamma}_1(s)|x_i(s)|ds + \widehat{M}_1.$$

By Grownwall's Inequality (see [6, Lemma 18.1.i]) we get

$$|x_i(t)| \leq \widehat{M}_1 \exp\Big(\int_0^1 \hat{\gamma}_1(s)ds\Big) := \widehat{M}_2. \tag{44}$$

Hence $\{x_i\}$ is bounded in $C([0,1],\mathbb{R}^n)$. From this and (43), we obtain

$$|\dot{x}_i| \leq \widehat{M}_2\hat{\gamma}_1(t) + \hat{\gamma}_2(t). \tag{45}$$

Hence

$$\|\dot{x}_i\|_1 \leq \widehat{M}_2\|\hat{\gamma}_1\|_1 + \|\hat{\gamma}_2\|_1. \tag{46}$$

Besides, if E is a measurable set of $[0,1]$, then form (45), we have

$$\int_E |\dot{x}_i(t)|dt \leq \widehat{M_2}\int_E \hat{\gamma}_1(t)dt + \int_E \hat{\gamma}_2(t)dt. \tag{47}$$

It is clear that the right hand side of (47) approaches to 0 as $|E| \to 0$. Hence $\{\dot{x}_j\}$ is equiabsolutely integrable and so $\{x_i\}$ is equiabsolutely continuous (see [6, Theorem 10.2.i, p. 317]). By Ascoli's Theorem, $\{x_i\}$ is a relatively compact set in $C([0,1],\mathbb{R}^n)$. Repeating the procedure as in the proof of (i), we find out an element $\overline{x} \in W^{1,1}([0,1],\mathbb{R}^n)$ such that $x_i \to \overline{x}$ uniform in $[0,1]$ and $\dot{x}_i \rightharpoonup \dot{\overline{x}}$ weakly in $L^1([0,1],\mathbb{R}^n)$. Also, since $\{u_i\}$ is bounded in $L^p([0,1],\mathbb{R}^m)$, there exists a subsequence u_{i_j} of $\{u_i\}$ such that $u_{i_j} \rightharpoonup \overline{u}$ for some $\overline{u} \in L^p([0,1],\mathbb{R}^m)$ as $j \to \infty$. By Lemma 3.3, we obtain $(\overline{x},\overline{u}) \in K(\overline{\lambda})$.

Let us claim that $(\overline{x},\overline{u}) \in S(\overline{\mu},\overline{\lambda})$. In fact, take any $(y,v) \in K(\overline{\lambda})$. By Lemma 3.2, we have

$$K(\overline{\lambda}) \subset K(\lambda_{i_j}) + k_0\|\lambda_{i_j} - \overline{\lambda}\|_s B_Z.$$

Hence there exists a sequence $(y_{i_j}, v_{i_j}) \in K(\lambda_{i_j})$ such that

$$\|y_{i_j} - y\|_{1,1} + \|v_{i_j} - v\|_p \leq k_0\|\lambda_{i_j} - \overline{\lambda}\|_s.$$

This implies that $y_{i_j} \to y$ in X and $v_{i_j} \to v$ in U. Since $(x_{i_j}, u_{i_j}) \in S(\mu_{i_j}, \lambda_{i_j})$, we have

$$J(x_{i_j}, u_{i_j}, \mu_{i_j}) = \int_0^1 f(t, x_{i_j}(t), u_{i_j}(t), \mu_{i_j}(t))dt \leq J(y_{i_j}, v_{i_j}, \mu_{i_j}) = \int_0^1 f(t, y_{i_j}(t), v_{i_j}(t), \mu_{i_j}(t))dt. \tag{48}$$

By $(H1), (H2)$ and $(H3)$, J is weakly lower semicontinuous (see [6, Theorem 10.8.i and Theorem 10.9.vii] and [7, Theorem 3.3, p. 84]), that is,

$$J(\overline{x},\overline{u},\overline{\mu}) \leq \liminf_{j\to\infty} J(x_{i_j}, u_{i_j}, \mu_{i_j}). \tag{49}$$

By $(H1)$, we have $f(t, y_{i_j}(t), v_{i_j}(t), \mu_{i_j}(t)) \to f(t, y(t), v(t), \overline{\mu}(t))$ a.e. $t \in [0,1]$. Since $y_{i_j} \to y$ uniformly on $[0,1]$, there exists a constant $M > 0$ such that $|y_{i_j}(t)| \leq M$ for all $t \in [0,1]$ and $j \geq 1$. Since $v_{i_j} \to v$ and $\mu_{i_j} \to \overline{\mu}$ strongly, there exist vector functions $v_0 \in L^p([0,1],\mathbb{R}^m)$ and $\mu_0 \in L^r([0,1],\mathbb{R}^k)$ such that

$$|v_{i_j}(t)| \leq |v_0(t)|,\ |\mu_{i_j}(t)| \leq |\mu_0(t)|$$

for all j and a.e. $t \in [0,1]$. Therefore, from $(H3)$ we have

$$|f(t, y_{i_j}(t), v_{i_j}(t), \mu_{i_j}(t))| \leq \vartheta(t) + \alpha_1 M^\alpha + \alpha_2|v(t)|^p + \alpha_3|\mu_0(t)|^r.$$

The Dominated Convergence Theorem implies that

$$\lim_{j\to\infty} J(y_{i_j}, v_{i_j}, \mu_{i_j}) = \int_0^1 f(t, y(t), v(t), \overline{\mu}(t))dt = J(y, v, \overline{\mu}). \tag{50}$$

Taking the limit on both sides of (48) and using (49) and (50), we get

$$J(\overline{x},\overline{u},\overline{\mu}) \le J(y,v,\overline{\mu}).$$

Since (y,v) is arbitrary in $K(\overline{\lambda})$, we get $(\overline{x},\overline{u}) \in S(\overline{\mu},\overline{\lambda}) \subset V$. The lemma is justified. □

• **Proof of Theorem 2.2**

Let V_1' be an open set in $C([0,1],\mathbb{R}^n)$ and V_2' be an open set in $L^p([0,1],\mathbb{R}^m)$ such that

$$S(\overline{\mu},\overline{\lambda}) \subset V_1' \times V_2' := V'. \tag{51}$$

We want to show that there exists a neighborhood $M_0 \times \Lambda_0$ of $(\overline{\mu},\overline{\lambda})$ such that

$$S(\mu,\lambda) \subset V', \forall(\mu,\lambda) \in M_0 \times \Lambda_0. \tag{52}$$

By contradiction, we find out a sequence $(\mu_i,\lambda_i) \to (\overline{\mu},\overline{\lambda})$ strongly in $L^\infty([0,1],\mathbb{R}^k) \times L^s([0,1],\mathbb{R}^l)$ and a sequence $(x_i,u_i) \in S(\mu_i,\lambda_i)$ such that $(x_i,u_i) \notin V'$. By Lemma 4.1, there exists $(\overline{x},\overline{u}) \in S(\overline{\mu},\overline{\lambda})$ and a subsequence $\{(x_{i_j},u_{i_j})\}$ of $\{(x_i,u_i)\}$ such that $x_{i_j} \to \overline{x}$ uniformly and $u_{i_j} \rightharpoonup \overline{u}$ weakly in $L^p([0,1],\mathbb{R}^m)$. If we can show that $u_{i_j} \to \overline{u}$ strongly then $(x_{i_j},u_{i_j}) \in V'$ for j large enough. This leads to a contradiction and so the theorem is proved. In the sequel, we shall denote by $\{(x_j,u_j)\}$ and $\{(\mu_j,\lambda_j)\}$ the subsequences $\{(x_{i_j},u_{i_j})\}$ and $\{(\mu_{i_j},\lambda_{i_j})\}$, respectively. It remains to prove the following lemma.

Lemma 4.2 *The sequence $\{u_j\}$ converges strongly to $\overline{u}$ in $L^p([0,1],\mathbb{R}^m)$.*

Proof. Since $(x_j,u_j) \in S(\mu_j,\lambda_j)$ and $(\overline{x},\overline{u}) \in S(\overline{\mu},\overline{\lambda})$, they must satisfy the Pontryagin principle. According to the Pontryagin Maximum Principle (see [10, Theorem 1, p. 134 and p. 139] and [3]), there exist absolutely continuous functions ϕ_j and $\overline{\phi}$ such that the following conditions are fulfilled:

$$\dot{\phi}_j(t)^T = -\phi_j(t)^T\big(A_x(t,x_j(t)) + B_x(t,x_j(t))u_j(t)\big) + f_x(t,x_j(t),u_j(t),\mu_j(t)),\ \ \phi_j(1)=0, \tag{53}$$

$$\dot{\overline{\phi}}(t)^T = -\overline{\phi}(t)^T\big(A_x(t,\overline{x}(t)) + B_x(t,\overline{x}(t))\overline{u}(t)\big) + f_x(t,\overline{x}(t),\overline{u}(t),\overline{\mu}(t)),\ \ \overline{\phi}(1)=0 \tag{54}$$

and for a.e. $t \in [0,1]$,

$$\begin{aligned} & f(t,x_j(t),u_j(t),\mu_j(t)) - \phi_j(t)^T(A(t,x_j(t)) + B(t,x_j(t)u_j(t)) \\ &= \min_{v\in[a(t),b(t)]}\{f(t,x_j(t),v,\mu_j(t)) - \phi_j(t)^T(A(t,x_j(t)) + B(t,x_j(t))v)\}, \end{aligned} \tag{55}$$

$$\begin{aligned} & f(t,\overline{x}(t),\overline{u}(t),\mu_j(t)) - \overline{\phi}(t)^T(A(t,\overline{x}(t)) + B(t,\overline{x}(t))\overline{u}(t)) \\ &= \min_{v\in[a(t),b(t)]}\{f(t,\overline{x}(t),v,\overline{\mu}(t)) - \overline{\phi}(t)^T(A(t,\overline{x}(t)) + B(t,\overline{x}(t))v)\}. \end{aligned} \tag{56}$$

Let us claim that $\phi_j - \overline{\phi} \to 0$ uniformly on $[0,1]$. Indeed, from (53) and (54), we have

$$
\begin{aligned}
&\dot{\phi}_j(t)^T - \dot{\overline{\phi}}(t)^T \\
&= -(\phi_j(t)^T - \overline{\phi}(t)^T)A_x(t,x_j) - \overline{\phi}(t)^T(A_x(t,x_j) - A_x(t,\overline{x})) \\
&- (\phi_j(t)^T - \overline{\phi}(t)^T)B_x(t,x_j)u_j - \overline{\phi}(t)^T(B_x(t,x_j)u_j - B_x(t,\overline{x})\overline{u}) \\
&+ f_x(t,x_j,u_j,\mu_j) - f_x(t,x_j,u_j,\overline{\mu}) + f_x(t,x_j,u_j,\overline{\mu}) - f_x(t,\overline{x},\overline{u},\overline{\mu}) \\
&= -(\phi_j(t)^T - \overline{\phi}(t)^T)A_x(t,x_j) - \overline{\phi}(t)^T(A_x(t,x_j) - A_x(t,\overline{x})) \\
&- (\phi_j(t)^T - \overline{\phi}(t)^T)B_x(t,x_j)u_j - \overline{\phi}(t)^T\big(B_x(t,x_j) - B_x(t,\overline{x})\big)u_j - \overline{\phi}(t)^T B_x(t,\overline{x})(u_j - \overline{u}) \\
&+ f_x(t,x_j,u_j,\mu_j) - f_x(t,x_j,u_j,\overline{\mu}) + f_x(t,x_j,u_j,\overline{\mu}) - f_x(t,\overline{x},\overline{u},\overline{\mu}).
\end{aligned}
$$

Define $\varphi_j(s) = \phi_j(1-s)$ and $\overline{\varphi}(s) = \overline{\phi}(1-s)$ with $s \in [0,1]$, we have $\frac{d}{ds}\varphi_j(s) = -\dot{\phi}_j(1-s)$ and $\varphi(0) = 0 = \overline{\varphi}(0)$. Moreover, from above we get

$$
\begin{aligned}
&-(\frac{d}{ds}\varphi_j(s)^T - \frac{d}{ds}\overline{\varphi}(s)^T) \\
&= -(\varphi_j(s)^T - \overline{\varphi}(s)^T)A_x(1-s,x_j) - \overline{\varphi}(s)^T(A_x(1-s,x_j) - A_x(1-s,\overline{x})) \\
&- (\varphi_j(s)^T - \overline{\varphi}(s)^T)B_x(1-s,x_j)u_j - \overline{\varphi}(s)^T\big(B_x(1-s,x_j) - B_x(1-s,\overline{x})\big)u_j \\
&- \overline{\varphi}(s)^T B_x(1-s,\overline{x})(u_j - \overline{u}) \\
&+ f_x(1-s,x_j,u_j,\mu_j) - f_x(1-s,x_j,u_j,\overline{\mu}) + f_x(1-s,x_j,u_j,\overline{\mu}) - f_x(1-s,\overline{x},\overline{u},\overline{\mu}).
\end{aligned}
$$

From this and

$$\varphi_j(s)^T - \overline{\varphi}(s)^T = \int_0^s (\frac{d}{ds}\varphi_j(\tau)^T - \frac{d}{ds}\overline{\varphi}^T(\tau))d\tau,$$

we get

$$
\begin{aligned}
&|\varphi_j(s) - \overline{\varphi}(s)| = |\varphi_j(s)^T - \overline{\varphi}(s)^T| = \Big|\int_0^s (\frac{d}{ds}\varphi_j(\tau)^T - \frac{d}{ds}\overline{\varphi}^T(\tau))d\tau\Big| \\
&\le \Big|\int_0^s (\varphi_j(\tau)^T - \overline{\varphi}(\tau)^T)A_x(1-\tau,x_j)d\tau\Big| + \Big|\int_0^s \overline{\varphi}(\tau)^T(A_x(1-\tau,x_j) - A_x(1-\tau,\overline{x}))d\tau\Big| \\
&+ \Big|\int_0^s (\varphi_j^T(\tau) - \overline{\varphi}^T(\tau))B_x(1-\tau,x_j)u_j d\tau\Big| + \Big|\int_0^s \overline{\varphi}(s)^T\big(B_x(1-\tau,x_j) - B_x(1-\tau,\overline{x})\big)u_j d\tau\Big| \\
&+ \Big|\int_0^s \overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j - \overline{u})d\tau\Big| + \Big|\int_0^s f_x(1-\tau,x_j,u_j,\mu_j) - f_x(1-\tau,x_j,u_j,\overline{\mu})d\tau\Big| \\
&+ \Big|\int_0^s f_x(1-\tau,x_j,u_j,\overline{\mu}) - f_x(1-\tau,\overline{x},\overline{u},\overline{\mu})d\tau\Big| \\
&\le \int_0^s |\varphi_j(\tau) - \overline{\varphi}(\tau))|\big(|A_x(1-\tau,x_j)| + |B_x(1-\tau,x_j)u_j|\big)d\tau \\
&+ \|\overline{\varphi}\|_0 \int_0^1 |A_x(1-\tau,x_j) - A_x(1-\tau,\overline{x})|d\tau + \|\overline{\varphi}\|_0 \int_0^1 |B_x(1-\tau,x_j) - B_x(1-\tau,\overline{x})||u_j|d\tau \\
&+ \sup_{s\in[0,1]} \Big|\int_0^s \overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j - \overline{u})d\tau\Big| + \int_0^1 |f_x(1-\tau,x_j,u_j,\mu_j) - f_x(1-\tau,x_j,u_j,\overline{\mu})|d\tau \\
&+ \int_0^1 |f_x(1-\tau,x_j,u_j,\overline{\mu}) - f_x(1-\tau,\overline{x},\overline{u},\overline{\mu})|d\tau. \qquad (57)
\end{aligned}
$$

Note that

$$\sup_{s\in[0,1]} \Big| \int_0^s \overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j-\overline{u})d\tau\Big| \le \int_0^1 |\overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})||(u_j-\overline{u})|d\tau$$
$$\le \|\overline{\varphi}^T B_x(\cdot,\overline{x})\|_q^q \|u_j-\overline{u}\|_p^p$$
$$\le \|\overline{\varphi}^T B_x(\cdot,\overline{x})\|_q^q M$$

for some constant $M>0$. Here we used the fact that $\{u_j-\overline{u}\}$ is bounded because $u_j \rightharpoonup \overline{u}$.

Since $u_j \rightharpoonup \overline{u}$, $x_j \to \overline{x}$ and $\mu_j \to \overline{\mu}$ uniformly, there exist positive numbers $\gamma_1, \gamma_2, \gamma_3$ such that

$$\|u_j\|_{L^p} \le \gamma_1,\ \|x_j\|_0 \le \gamma_2, \quad \|\mu_j\|_\infty \le \gamma_3, \quad \forall j \ge 1.$$

Since k_i is continuous, we obtain

$$k_i(t,|x_j(t)|,|\mu_j(t)|,|\overline{\mu}(t)|) \le \xi_i := \max_{(t_1,t_2,t_3,t_4)\in[0,1]\times[0,\gamma_2]\times[0,\gamma_3]\times[0,\|\overline{\mu}\|_\infty]} k_i(t_1,t_2,t_3,t_4) \tag{58}$$

with $i=1,2$. Combining this with (9) and (11), we have

$$\begin{aligned}
&\int_0^1 |f_x(1-\tau,x_j,u_j,\mu_j) - f_x(1-\tau,x_j,u_j,\overline{\mu})|d\tau \\
&+ \int_0^1 |f_x(1-\tau,x_j,u_j,\overline{\mu}) - f_x(1-\tau,\overline{x},\overline{u},\overline{\mu})|d\tau \\
&\le \int_0^1 \xi_1 |u_j|^\eta |\mu_j(1-\tau)-\overline{\mu}(1-\tau)|^{s_1} d\tau + \int_0^1 k_3(1-\tau)|x_j(1-\tau)-\overline{x}(1-\tau)|d\tau \\
&\le C_1\xi_1 \|u_j\|_{L^p}^\eta \|\mu_j-\overline{\mu}\|_{L^\infty}^{s_1} + \|k_3(\cdot)\|_{L^1}\|x_j-\overline{x}\|_0 \\
&\le C_1\xi_1\gamma_1^\eta \|\mu_j-\overline{\mu}\|_{L^\infty}^{s_1} + \|k_3(\cdot)\|_{L^1}\|x_j-\overline{x}\|_0
\end{aligned}$$

for some constant $C_1>0$. From this and (57), we get

$$\begin{aligned}
|\varphi_j(s)-\overline{\varphi}(s)| &\le \int_0^s |\varphi_j(\tau)-\overline{\varphi}(\tau))|\big(|A_x(1-\tau,x_j(\tau))| + |B_x(1-\tau,x_j)u_j|\big)d\tau \\
&+ \|\overline{\varphi}\|_0 \int_0^1 |A_x(1-\tau,x_j) - A_x(1-\tau,\overline{x})|d\tau + \|\overline{\varphi}\|_0 \int_0^1 |B_x(1-\tau,x_j)-B_x(1-\tau,\overline{x})||u_j|d\tau \\
&+ \sup_{s\in[0,1]}\Big|\int_0^s \overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j-\overline{u})d\tau\Big| + C_1\xi_1\gamma_1^\eta\|\mu_j-\overline{\mu}\|_{L^\infty}^{s_1} + \|k_3(\cdot)\|_{L^1}\|x_j-\overline{x}\|_0.
\end{aligned}$$

By Gronwall's Inequality for integral form, we obtain

$$\begin{aligned}
&|\varphi_j(s)-\overline{\varphi}(s)| \\
&\le \exp\Big(\int_0^1 (|A_x(1-\tau,x_j)| + |B_x(1-\tau,x_j)u_j|)d\tau\Big)\Big\{\|\overline{\varphi}\|_0\int_0^1 |A_x(1-\tau,x_j)-A_x(1-\tau,\overline{x})|d\tau \\
&+ \|\overline{\varphi}\|_0 \int_0^1 |B_x(1-\tau,x_j)-B_x(1-\tau,\overline{x})||u_j|d\tau + \sup_{s\in[0,1]}\Big|\int_0^s \overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j-\overline{u})d\tau\Big| \\
&+ C_1\xi_1\gamma_1^\eta \|\mu_j-\overline{\mu}\|_{L^\infty}^{s_1} + \|k_3(\cdot)\|_{L^1}\|x_j-\overline{x}\|_0\Big\}.
\end{aligned} \tag{59}$$

Let us show that the right hand side of (59) converges to 0 as $j \to \infty$. Note that since $A_x(\cdot,\cdot)$ and $B_x(\cdot,\cdot)$ are continuous and $\|x_j\|_0 \le \gamma_2$, we have

$$|A_x(t, x_j(t))| \le \sup_{(t,x)\in[0,1]\times\gamma_2 B_n} |A_x(t,x)| < +\infty \tag{60}$$

$$|B_x(t, x_j(t))| \le \sup_{(t,x)\in[0,1]\times\gamma_2 B_n} |B_x(t,x)| < +\infty, \tag{61}$$

where B_n is the unit ball in $\mathbb{R}^n$. We have

$$\begin{aligned}\int_0^1 (|A_x(1-\tau, x_j)| + |B_x(1-\tau, x_j)u_j|)d\tau &\le \int_0^1 (|A_x(1-\tau, x_j)|d\tau + \|B_x(\cdot, x_j)\|_q \|u_j\|_p \\ &\le \int_0^1 (|A_x(1-\tau, x_j)|d\tau + \|B_x(\cdot, x_j)\|_q \gamma_1. \quad (62)\end{aligned}$$

From (60), (61) and the Dominated Convergence Theorem, we see that the right hand side of (62) converges to $\int_0^1 (|A_x(1-\tau,\overline{x})|d\tau + \|B_x(\cdot,\overline{x})\|_q\gamma_1$ and so it is bounded. Hence

$$\int_0^1 (|A_x(1-\tau, x_j)| + |B_x(1-\tau, x_j)u_j|)d\tau \le M_1, \ \forall j \ge 1$$

for some constant $M_1 > 0$. Also, by the Dominated Convergence Theorem, we have

$$\|\overline{\varphi}\|_0 \int_0^1 |A_x(1-\tau, x_j) - A_x(1-\tau,\overline{x})|d\tau + \|\overline{\varphi}\|_0 \int_0^1 |B_x(1-\tau, x_j) - B_x(1-\tau,\overline{x})||u_j|d\tau \to 0$$

as $j \to \infty$. The last term in (59) also converges to 0 because $\mu_j \to \overline{\mu}$ and $x_j \to \overline{x}$ uniformly. We now show that

$$\sup_{s\in[0,1]} \Big| \int_0^s \overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j - \overline{u})d\tau \Big| \to 0 \text{ as } j \to \infty. \tag{63}$$

By contradiction, there exists $\epsilon_1 > 0$ such that

$$\sup_{s\in[0,1]} \Big| \int_0^s \overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j - \overline{u})d\tau \Big| > \epsilon_1, \ \forall j \ge 1.$$

Hence for each j, there exist $s_j \in [0,1]$ such that

$$\Big| \int_0^{s_j} \overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j - \overline{u})d\tau \Big| > \epsilon_1, \ \forall j \ge 1.$$

By passing to a subsequence if necessary, we can assume that $s_j \to s_0 \in [0,1]$. From the above, we have

$$\begin{aligned}\epsilon_1 &< \Big| \int_0^{s_j} \overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j - \overline{u})d\tau \Big| \\ &\le \Big| \int_0^{s_0} \overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j - \overline{u})d\tau \Big| + \Big| \int_{s_0}^{s_j} \overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j - \overline{u})d\tau \Big| \\ &\le \Big| \int_0^1 1_{[0,s_0]}(\tau)\overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})(u_j - \overline{u})d\tau \Big| \\ &+ \Big(\int_{s_0}^{s_j} |\overline{\varphi}(\tau)^T B_x(1-\tau,\overline{x})|^q d\tau \Big)^{1/q} \|u_j - \overline{u}\|_p^p, \end{aligned} \tag{64}$$

where $1_{[0,s_0]}$ is the indicator function of interval $[0, s_0]$. It is easy to see that

$$1_{[0,s_0]}(\cdot)\overline{\varphi}(\cdot)^T B_x(1-\cdot, \overline{x}) \in L^q([0,1], \mathbb{R}^m).$$

Since $u_j \rightharpoonup \overline{u}$ weakly in $L^p([0,1], \mathbb{R}^m)$, we get

$$\Big|\int_0^1 1_{[0,s_0]}(\tau)\overline{\varphi}(\tau)^T B_x(1-\tau, \overline{x})(u_j - \overline{u})d\tau\Big| \to 0 \text{ as } j \to \infty.$$

Since $\|u_j - \overline{u}\|_p^p$ is bounded and $|\overline{\varphi}(\tau)^T B_x(1-\tau, \overline{x})|$ is continuous, we get

$$\Big(\int_{s_0}^{s_j} |\overline{\varphi}(\tau)^T B_x(1-\tau, \overline{x})|^q d\tau\Big)^{1/q} \|u_j - \overline{u}\|_p^p \to 0 \text{ as } j \to \infty.$$

By letting $j \to \infty$ in (64), we obtain a contradiction. Hence (63) is valid.

In summary, we have shown that the right hand side of (59) converges to 0 as $j \to \infty$. Consequently, $\varphi_j \to \overline{\varphi}$ uniformly. Hence $\phi_j \to \overline{\phi}$ uniformly on $[0, 1]$. The claim is justified.

From (55) and (56), we see that u_j and $\overline{u}$ satisfy variational inequalities

$$\langle f_u(t, x_j(t), u_j(t), \mu_j(t)) - \phi_j(t)^T B(t, x_j(t)), v - u_j(t)\rangle \geq 0 \ \forall v \in [a(t), b(t)]$$

and

$$\langle f_u(t, \overline{x}(t), \overline{u}(t), \overline{\mu}(t)) - \overline{\phi}(t)^T B(t, \overline{x}(t)), v - \overline{u}(t)\rangle \geq 0 \ \forall v \in [a(t), b(t)],$$

respectively. Hence

$$\langle f_u(t, x_j(t), u_j(t), \mu_j(t)) - \phi_j(t)^T B(t, x_j(t)), \overline{u}(t) - u_j(t)\rangle \geq 0$$

and

$$\langle f_u(t, \overline{x}(t), \overline{u}(t), \overline{\mu}(t)) - \overline{\phi}(t)^T B(t, \overline{x}(t)), u_j(t) - \overline{u}(t)\rangle \geq 0$$

for a.e. $t \in [0, 1]$. Using above inequalities and (12), we get

$$\begin{aligned}
&\alpha|u_j(t) - \overline{u}(t)|^p \leq \langle f_u(t, x_j(t), u_j(t), \overline{\mu}(t)) - f_u(t, \overline{x}(t), \overline{u}(t), \overline{\mu}(t)), u_j(t) - \overline{u}(t)\rangle \\
&\leq \langle f_u(t, x_j(t), u_j(t), \overline{\mu}(t)) - f_u(t, \overline{x}(t), \overline{u}(t), \overline{\mu}(t)), u_j(t) - \overline{u}(t)\rangle \\
&+ \langle f_u(t, x_j(t), u_j(t), \mu_j(t)) - \phi_j(t)^T B(t, x_j(t)), \overline{u}(t) - u_j(t)\rangle \\
&+ \langle f_u(t, \overline{x}(t), \overline{u}(t), \overline{\mu}(t)) - \overline{\phi}(t)^T B(t, \overline{x}(t)), u_j(t) - \overline{u}(t)\rangle \\
&= \langle f_u(t, x_j(t), u_j(t), \overline{\mu}(t)) - f_u(t, x_j(t), u_j(t), \mu_j(t)), u_j(t) - \overline{u}(t)\rangle \\
&+ \langle \phi_j(t)^T(t) B(t, x_j(t)) - \overline{\phi}(t)^T B(t, \overline{x}(t)), u_j(t) - \overline{u}(t)\rangle \\
&\leq |f_u(t, x_j(t), u_j(t), \overline{\mu}(t)) - f_u(t, x_j(t), u_j(t)||u_j(t) - \overline{u}(t)| \\
&+ |\phi_j(t)^T B(t, x_j(t)) - \overline{\phi}(t)^T B(t, \overline{x}(t))||u_j(t) - \overline{u}(t)|.
\end{aligned}$$

It follows that for a.e. $t \in [0,1]$,

$$\begin{aligned}\alpha|u_j(t)-\overline{u}(t)|^{p-1} &\le |f_u(t,x_j(t),u_j(t),\overline{\mu}(t)) - f_u(t,x_j(t),u_j(t)| \\ &\quad + |\phi_j(t)^T B(t,x_j(t)) - \overline{\phi}(t)^T B(t,\overline{x}(t))|.\end{aligned}$$

Combining this with (10) and (58), we get

$$\alpha|u_j(t)-\overline{u}(t)|^{p-1} \le \xi_2|u_j(t)|^{\theta}|\mu_j-\overline{\mu}|^{s_2} + |\phi_j(t)^T B(t,x_j(t)) - \overline{\phi}(t)^T B(t,\overline{x}(t))|.$$

Using the inequality $(a+b)^q \le 2^{q-1}(a^q+b^q)$ for $a,b \ge 0$ and $q \ge 1$ (see [1, Lemma 2.24, p. 34]), yields

$$\begin{aligned}&\alpha^q|u_j(t)-\overline{u}(t)|^{q(p-1)} = \alpha^q|u_j(t)-\overline{u}(t)|^p \\ &\le 2^{q-1}\big(\xi_2^q\|u_j(t)|^{\theta q}|\mu_j(t)-\overline{\mu}(t)|^{s_2 q} + |\phi_j(t)^T B(t,x_j(t)) - \overline{\phi}(t)^T B(t,\overline{x}(t))|^q\big).\end{aligned}$$

Here we used the equality $q(p-1)=p$. Integrating on $[0,1]$ and using the facts $\theta q \le p$ and $\|u_j\|_p \le \gamma_1$, we obtain

$$\alpha^q\|u_j-\overline{u}\|^p_{L^p} \le 2^{q-1}\Big(C_2\|\mu_j-\overline{\mu}\|^{s_2 q}_{L^\infty}\gamma_1^{\theta q} + \int_0^1 |\phi_j(t)^T B(t,x_j(t)) - \overline{\phi}(t)^T B(t,\overline{x}(t))|^q dt\Big) \tag{65}$$

for some absolutely constant $C_2 > 0$. Since $|\phi_j(t)^T B(t,x_j(t)) - \overline{\phi}(t)^T B(t,\overline{x}(t))| \to 0$ and (7), the Dominated Convergence Theorem implies that

$$\int_0^1 |\phi_j(t)^T B(t,x_j(t)) - \overline{\phi}(t)^T B(t,\overline{x}(t))|^q dt \to 0 \text{ as } j \to \infty.$$

Combining this with the fact that $\mu_j \to \overline{\mu}$ in $L^\infty([0,1],\mathbb{R}^l)$, we see that the right hand side of (65) converges to 0 as $j \to \infty$. Hence $u_j \to \overline{u}$ strongly in $L^p([0,1],\mathbb{R}^m)$. The lemma is proved.

Finally, if $S(\overline{\mu},\overline{\lambda})$ is a singleton, then $S(\cdot,\cdot)$ is lower semicontinuous at $(\overline{\mu},\overline{\lambda})$. In fact, let V_1 be an open set in $C([0,1],\mathbb{R}^n)$ and V_2 be an open set in $L^p([0,1],\mathbb{R}^m)$ such that $S(\overline{\mu},\overline{\lambda}) \cap (V_1 \times V_2) \ne \emptyset$. Since $S(\overline{\mu},\overline{\lambda}) = \{(\overline{x},\overline{u})\}$, we have $S(\overline{\mu},\overline{\lambda}) \subset (V_1 \times V_2)$. By upper semicontinuity of $S(\cdot,\cdot)$ at $(\overline{\mu},\overline{\lambda})$, there are neighborhoods U_1 of $\overline{\mu}$ and U_2 of $\overline{\lambda}$ such that $S(\mu,\lambda) \subset V_1 \times V_2$ for all $(\mu,\lambda) \in U_1 \times U_2$ and so $S(\mu,\lambda) \cap (V_1 \times V_2) \ne \emptyset$ for all $(\mu,\lambda) \in U_1 \times U_2$. Hence $S(\cdot,\cdot)$ is $(s,s)-$lower semicontinuous at $(\overline{\mu},\overline{\lambda})$. This implies that $S(\cdot,\cdot)$ is continuous at $(\overline{\mu},\overline{\lambda})$. The proof of Theorem 2.2 is now complete. □

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